Math 400 paper 2
Matthew dreher

Graph theory matching:
Start with we need to talk about what is graph theory and the part of it that are conserved with matchings. Graph theory is the study of graphs, which are mathematical structures that are made up of points called vertices or node which can be connected by edges or lines trams chose of term is normal determined by if graph is undirected or not for my purposes I will only talk about undirected mean I will only use the terms vertex and edges. A vertex is adjacent if there exist a shared edge between the vertices, There is a couple of types graph of interests a one being the a connected graph which means that between any two vertices there exists a walk a sequence of adjacent vertices from vertex (A) to vertex (B) or path which is a walk with no repeated vertices in the walk, if there exist a walk then there exist a path the reverse is also true. Regular and complete and cycles, a graph is regular if each vertex's has the same number of degree (same number of edges) denoted by i-regular where $i$ is the degree of each vertices, cycles graph is a special type 2 -regular where it connected and therefore has a path from vertex (A) to (A), and complete graph which is a regular graph with is ( $\mathrm{n}-1$ )-regular where n is the number vertices in the graph commonly denoted by $\mathrm{k}_{\mathrm{i}}$ where I is the number of vertices in the graph this is also the maxim number of edges for a simple graph.

We define a Matching as a collection of edges such that no two edges are adjacent in a graph or a subset $M$ of $G, M=(V, E)$ where $V$ is the set vertices and $E$ is the set of edges such, that no two edges in $M$ share a common vertex, if $M$ has the largest size among all of the matching in $G$ then it's called a maximum matching, the maximum size $M<=n / 2$, a special case is when the $M$ is size $n / 2$ edge which is called a perfect matching it is a necessary condition for the graph to have a even number of vertices since there does not exist a 1-regluar graph with odd number of vertices due to the handshaking theorem which states that the number of edges is half of the sum of the degree of the graph. One important Theorem used to help find the maximum matching is the Berge's Theorem which states that a matching $M$ is maximum if and only if there is no augmenting path with respect to M , an augmenting path is an alternating path
between two free vertices and a alternately path is a path whose edges are alternately in and out of the matching. proof let M be a matching that is not the maximum matching then there exists an $M^{\prime}$ such that $M^{\prime}>M$ then consider the set of edges in $M^{\prime}$ not in $M$ since $M^{\prime}$ and $M$ are matching the degree of each vertices is at most 2 due to the fact that each vertices has at most has a degree 1 let $H=M$ union $M^{\prime}$, so $H$ is a collection of isolated vertex an even cycle whose edges alternate between M and $\mathrm{M}^{\prime}$ a path with distant end point and with edges alternate between $M$ and $M^{\prime}$ Since $M^{\prime}$ is larger than $M$, there contains a component that has more edges from $M^{\prime}$ than $M$. Such a component is a path in $G$ that starts and ends with an edge from $M^{\prime}$, so it is an augmenting path. Using the Blossom algorithm one can find the maximum matching a commonly used example of pseudocode for the blossom algorithm

```
INPUT: Graph G, initial matching }M\mathrm{ on }
OUTPUT: maximum matching }\mp@subsup{M}{}{*}\mathrm{ on }
function find_maximum_matching(G, M) : M*
    P}\leftarrow\mathrm{ find_augmenting_path(G, M)
    if }P\mathrm{ is non-empty then
        return find_maximum_matching(G, augment }M\mathrm{ along P)
    else
        return M
    end if
end function
Graph G, matching }M\mathrm{ on }
OUTPUT: augmenting path P in G or empty path if none found
function find_augmenting_path(G, M) : P
    F}\leftarrow\mathrm{ empty forest
    unmark all vertices and edges in G, mark all edges of M
    for each exposed vertex }V\mathrm{ do
        create a singleton tree { V } and add the tree to F
    end for
    while there is an unmarked vertex v in }F\mathrm{ with distance(v,
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root(v)) even do

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            while there exists an unmarked edge e = {v, w } do
            if }W\mathrm{ is not in }F\mathrm{ then
                        # w is matched, so add e and w's matched edge
to F
                    x v vertex matched to }W\mathrm{ in }
                            add edges { V, w } and { W, X } to the tree of V
        else
            if distance(w, root(w)) is odd then
                            # Do nothing.
            else
                if root (v) }=\mathrm{ root (w) then
                    # Report an augmenting path in F
{ e }.
```



```
root(w))
            return P
                else
                    # Contract a blossom in G and look
for the path in the contracted graph.
                            B}\leftarrow\mathrm{ blossom formed by e and edges on the
path V }V\mathrm{ w in T
                                    G',}\mp@subsup{M}{}{\prime}\leftarrow\mathrm{ contract }G\mathrm{ and }M\mathrm{ by }
                                    P'}\leftarrow\mathrm{ find_augmenting_path(G', M')
                            P}\leftarrow\mathrm{ lift }\mp@subsup{P}{}{\prime}\mathrm{ to }
                            return P
            end if
    end if
    end if
    mark edge e
end while
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        mark vertex v
    end while
    return empty path
end function
```

Another important type of graph is the bipartite graph is a type of graph that is made up of two set of vertices where each vertex in a set is not adjutant to another vertex in the set

Hall's Theorem "let $G$ be a bipartite graph with partite sets $U$ and $W$ such that $R|U|<=|W|$. then $G$ contains a matching of size $r$ if and only if $|N(X)|>=|X|$ for every nonempty subset $X$ of U."
"proof If every $\mathrm{k} \mathrm{y}^{\prime} \mathrm{s}$ (where $\mathrm{k}<\mathrm{m}$ ) collectively know at least $\mathrm{k}+1 \mathrm{x}$ 's, so that the condition is always true 'with one $x$ to spare', then we take any $y$ and pair to any $x$ 's with an edges with $y$ 's. The original condition then remains true for the other $m-y$, who can be paired by induction, completing the proof in this case.
(ii) If now there is a set of $k y^{\prime} s(k<m)$ who collectively know exactly $k x^{\prime} s$, then these $k y^{\prime} s$ can be paired by induction to the $k x^{\prime} s$, leaving $m-k y$ 's still to be paired. But any collection of $h$ of these $m-k y^{\prime} s$, for $h<=m-k$, must know at least $h$ of the remaining $x^{\prime} s$, since otherwise these $h y$ 's, together with the above collection of $k y$ 's, would collectively know fewer than $h+k$ edges with x's, contrary to our assumption. It follows that the original condition applies to the $m-k y$ 's. They can therefore be matched by induction."(Wilson).

Stable marriage problem is a problem asking given two groups and with the goal of matching each element of one group with the other group in such a way that no match ( $A, B$ ) exist that which both prefer each other to their current partner under the matching. The Algorithmic solution is each element creates a list order of prefers pairing select one group to make a graph of $A$ with there prefers pair if multicable $A$ match with a $B$ then pick the $A$ that is most prefers
by that $B$, for $A$ that does not pair have them try pairing with there next prefers pairing this is repeat until there is a perfect matching, one thing to note is that the match $A$ ask $B$ is not equal to B ask A

## Refences

- Benjamin, Arthur, et al. The Fascinating World of Graph Theory. Princeton University Press, 2015. JSTOR, www.jstor.org/stable/j.ctt9qhOpv. Accessed 2 Nov. 2020.
- Wilson, R. J. (2015). Introduction to graph theory. Harlow, United Kingdom: Prentice Hall.

