Fractals

Ruo Chen
The College of William and Mary

Sept. 22, 2020

1 Introduction

Geometry has always been an important topic in the field of mathematics. The geometry most people know of is called the Euclidean geometry, which is widely favored by mathematicians. In this paper, I aim to expose a completely different notion of geometry - fractal geometry. The paper is organized into five parts. In the first part, we will begin by outlining the history of fractal geometry. In the second part, we will elaborate the reason this topic is chosen and how it fits the goal of this course, which is to connect mathematics to other areas and disciplines. In the thrid part, we will dive deeper into fractal geometry and discuss the concept of fractal dimension. In this section, we will also provide some examples to help with the understanding of this abstract concept and elaborate on the importance of fractal dimension. In the forth part, we will present some examples of the real-world application of fractals and further discuss its interdisciplinary connections. Lastly, we will conclude the paper by talking about some possible topics for future research that were suggested by the class and Professor Li.

2 History Background

The reason we begin this paper with the history of fractal is that mathematical studies, at least on undergraduate level, usually only cover mathematical knowledge that had been discovered a few hundred years ago to help us build a solid foundation for higher level math. So in some sense, the math we are learning right now is analogous to history study for mathematicians. Now let us talk about the history of fractals.

Fractal, any of a class of complex geometric shapes that commonly have "fractional dimension", is a concept first introduced by the mathematician Felix Hausdorff in 1918. The term fractal, derived from the Latin word fractus ("fragmented," or "broken"), was coined by unconventional 20th century mathematician Benoit Mandelbrot in 1975. Benoit Mandelbrot was a Polish-born mathematician who studied math in Paris, France. He spent most of his life in the United States working for IBM. Mandelbrot was considered the father of fractal geometry and was largely responsible for the present interest in the subject. [3][5]

Mandelbrot did not invent or discover these shapes. In fact, mathematicians had been exploring the concept of fractals way before the term was even used, Gaston Julia devised the idea of using a feedback loop to produce a repeating pattern in the early 20th century. Georg Cantor experimented with properties of recursive and self-similar sets in the 1880s, and in 1904 Helge von Koch published the concept of an infinite curve, using approximately the same technique but with a continuous line. And Lewis Richardson explored Koch's idea while trying to measure English coastlines.

These explorations into such complex mathematics were mostly theoretical, however. Lacking at the time was a machine capable of performing the grunt work of so many mathematical calculations in a reasonable amount of time to find out where these ideas really led. As the power of computers evolved, so too did the ability of mathematicians to test these theories. [3]

3 Topic Selection

When deciding on the topics I would like to present for the first round of presentation, I wanted to present something unconventional, or at least something that is counter intuitive. I came across fractals while rewatching the movie “Doctor Strange” with my friends. The film used fractals to create sights that people had never seen in a movie before. As characters try to navigate bizarre changes to their reality, scenes zoom in or out on a building, wall or floor. And this reveals more buildings, walls and floors within. At the time, I was simply amazed by this work of art from the filmmakers and had no idea fractal was the magic behind all these other worldly visual experiences. Hence, I dove into fractals and it completely changed the way I understand geometry. When learning about fractals, I inevitably came across the father of fractal geometry, Benoit Mandelbrot, and his famous book *The Fractal Geometry of Nature*. The very first paragraph of the book described the motivation behind fractal geometry, “Why is geometry often described as ‘cold’ and ‘dry?’ One reason lies in its inability to describe the shape of a cloud, a mountain, a coastline, or a tree. Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line.” [2]

Regarding the theme of “mathematical connection”, I think this topic is very fitting as fractal geometry can be used to connect with the real world than Euclidean geometry, which is often referred to as standard geometry. The motivation behind fractal geometry is rather pragmatic than just creating some beautiful infinitely self-repeating shapes as Mandelbrot claimed that the purpose of fractal geometry is to help people model the nature without neglecting the finer details of the things they are actually modeling. Due to the complexity of the subject, the goal of my presentation is to briefly introduce the concept of fractal geometry to the class and discuss its application in different fields.

4 Basic Concepts of Fractal Geometry

4.1 Terminology

Before we get into any details, we need to cover some basic terminology that will help with the understanding of the unique qualities that fractals possess. All fractals show a degree of what’s called **self-similarity**. This means that as you look closer and closer into the details of a fractal, you can see a replica of the whole. We can use computer to generate some complex pattern using a simple equation or mathematical statements. Fractals are created by repeating this equation through a feedback loop in a process called **iteration**, where the results of one iteration form the input value for the next. The idea of iteration is heavily used in computer science and programmers are especially fond of utilizing iteration to automate repetitive tasks. Fractals are also **recursive**, regardless of scale. Imagine you are in a room surrounded by mirrors. What you are looking at in the mirror is an infinitely recursive image of yourself. Lastly, we need to understand that the geometry we grew up learning is most likely Euclidean geometry.[3] Dimension is something that usually only makes sense for real numbers. In fact, most people’s understanding of geometry stops at three dimensions, which are length, width, and height. However, fractal geometry completely abandoned this concept and created irregular shapes in fractal dimension. The fractal dimension of a shape is used to measure that shape’s complexity and we will dive deeper into the concept in the next section of this paper.

Now, a pure fractal is defined as a geometric shape that is perfectly self-similar through infinite iteration in a recursive pattern and through infinite detail. So how does that help us describe or model the nature that is full of irregular and fragmented shapes? One of the most common misconceptions is that fractals are shapes that are perfectly self-similar. Indeed, self-similar shapes are beautiful because of its infinite complexity and are good toy model for what fractals really are. But Mandelbrot had a much broader conception in mind, one motivated not by beauty but by a more pragmatic desire to model nature in a way that actually captures roughness. In some way, fractal geometry is a rebellion against calculus, whose central assumption is that things tend to look smooth if you zoom in far enough. But Mandelbrot thought this is overly idealized, resulting in models that neglect the finer details of the things they are actually modeling.

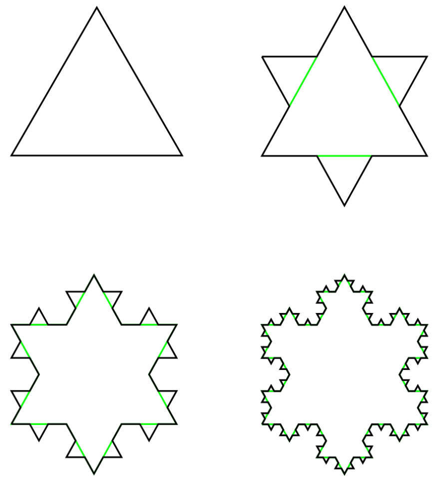

Figure 1: The Koch Snowflake with 4 iterations

One of Mandelbrot's early demonstration of fractals was similar to von Koch's snowflake (*Figure 1*). However, his realization of what fractals really are arose from a question: How long is the coastline of Great Britain? The question seems simple. The answer is not. Measure a coastline on a globe or from satellite images, and you can use a ruler to find the solution. But if you hop in a boat and follow the coastline all the way around, you'll get a larger number. because you can measure more twists and turns, which add distance. If you walk the whole length, you'll get a still bigger number. If you could enlist an ant to do the measurement for you, it will report an even bigger number. That's because it would have to scramble over or around every rock it encountered. Mandelbrot showed that the measured length depends on the size of your ruler. The smaller your ruler, the larger your answer. By that process, he said, the coastline is infinitely long.

Once the basic concept of a Fractal is understood, it is shocking to see how many unique types of Fractals exist in nature. Some of the most common examples of Fractals in nature would include branches of trees, snowflakes, lightning, plants and leaves and river systems, clouds, crystals. For example, we've all heard the saying that every snowflake is unique and one of the contributing factors to the uniqueness of snowflakes is that they form in Fractal patterns which can allow for incredible amounts of detail and also variation. In the case of ice crystal formations, the starting point of the Fractal is in the center and the shape expands outward in all directions. As the crystal expands, the Fractal structures are formed in each direction.[**?**]

4.2 Fractal Dimension

In general, dimension usually only make sense for natural numbers. We think of mountains and other objects in the real world as having three dimensions. In Euclidean geometry we assign values to an object's length, height and width, and we calculate attributes like area, volume and circumference based on those values. But most objects are not uniform. [1] The English coastline mentioned above, for example, have jagged edges. Fractal geometry enables us to more accurately define and measure the complexity of a shape by quantifying how rough its edges is. The jagged edges of the coastline can be expressed mathematically using the fractal dimension. Before diving into fractal dimensions, we have to have a basic understanding of how it is calculated.

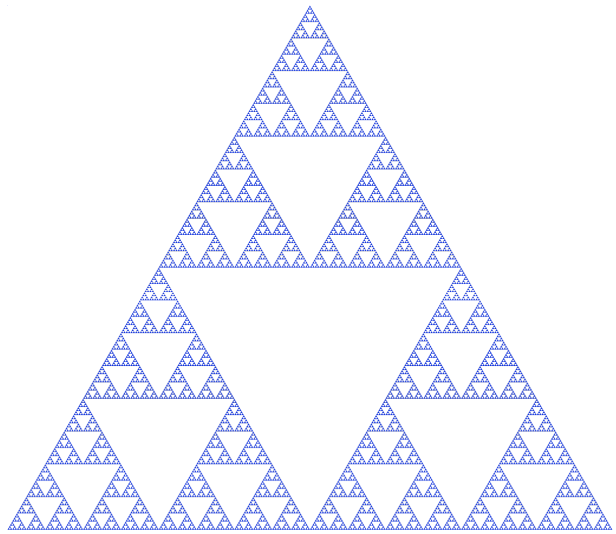

Figure 2: The Sierpinski triangle

We can start with four perfectly self-similar shapes, a line, a square, a cube, and a Sierpinski triangle (*Figure 2*), which is a fractal attractive fixed set with the overall shape of an equilateral triangle, subdivided recursively into smaller equilateral triangles. Note that the first three shapes are not fractals, however, they will help us understand how fractal dimensions work, at least on a very fundamental level.

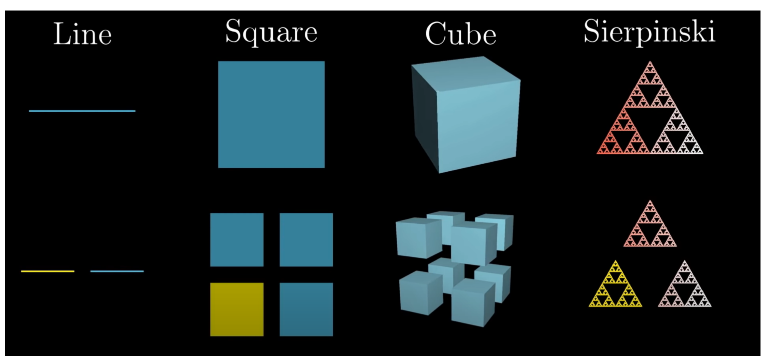

Figure 3: Four shapes that we will be using to demonstrate the concept of fractal dimension

As shown in (*Figure 3*), a line can be split into two identical copies of its own, scaled down by $1/2$. A square can be split into four identical squares scaled down by $1/2$, and a cube can be split into eight identical cubes, each is a scaled down version by $1/2$. The famous Sierpinski triangle consists of three similar identical copies of itself, scaled down by $1/2$ as well. So it is obvious that they are all perfectly self-similar. Now we have to think about how we measure these shapes. We say the smaller line has $1/2$ the length of the original line. The smaller square has $1/4$ the area of the original square. And the smaller cube has $1/8$ the volume of the original cube. But how about the Sierpinski triangle? If we take the Euclidean approach, which is calculating the perimeter and area of the Sierpinski triangle. Since the Sierpinski triangle is generated by subdividing an equilateral recursively into smaller equilateral triangles with infinite iterations, we will find that the perimeter of Sierpinski triangle is infinite and its area is zero as shown below:

Perimeter: The perimeter of Sierpinski triangle increases by a factor of $\frac{3}{2}$ in each iteration. Thus, we can express the total perimeter of the triangle as a function of number of iteration:

$$P = P_0 * \left(\frac{3}{2}\right)^n$$

where P_0 is the initial perimeter of the original equilateral triangle. From this expression we can see that the total perimeter length of a Sierpinski triangle is infinite. We can verify this by taking the limit of the perimeter function:

$$\lim_{n\to\infty} P_0 * \left(\frac{3}{2}\right)^n = \infty$$

Area: The area of the Sierpinski triangle can be expressed as a function of number of iteration n:

$$A = A_0 * \left(\frac{3}{4}\right)^n$$

whereA_0 is the initial area of the original equilateral triangle. Similar to the perimeter, we can see that the total area of a Sierpinski triangle is 0. Again, we can verify this by taking the limit of the area function:

$$\lim_{n\to\infty} A_0 * \left(\frac{3}{4}\right)^n = 0$$

Now we know that the Sierpinski triangle indeed has infinite perimeter and zero area. In this way, the Sierpinski Triangle is not one dimensional, although you could define a curve that passes through all its point. And nor is it two dimensional, even though it lives in the plane. So what is the dimensionality of the Sierpinski triangle?

In order to understand fractal dimension, we need to reconsider the notion of dimension. Length, area, and volume can be generalized as "measure". However, it would be a lot easier to understand fractal dimensions if we think about it in terms of "mass" instead of "measure". Intuitively, it does not seem to make sense mathematically. We will now explain why using mass would help us understand fractal dimension. Think of the line as a thin wire, the square as a flat sheet, the cube as a solid cube, and the Sierpinski triangle as some sort of Sierpinski mesh and they are all made of some material with uniformed density. As previously mentioned, each shape was scaled down by 1/2. As seen in *Figure 3*, the mass of the smaller line, or wire, would be 1/2 of that of the original line. So we can say the mass scaling factor for the line is 1/2. The mass of each smaller identical copy is 1/4 of that of the original square, which means its mass scaling factor is 1/4. Similarly, the mass scaling factor for the solid cube is 1/8. Lastly, each smaller copy of the original Sierpinski triangle has 1/3 the mass of the original triangle. We can sort of see a pattern here if we put the numbers into a table (*Figure 4*):

Shapes	**Line**	**Square**	**Cube**	**Sierpinski triangle**
Dimension	**1**	**2**	**3**	**?**
Scaling factor	**1/2**	**1/2**	**1/2**	**1/2**
Mass scaling factor	**1/2**	**1/4**	**1/8**	**1/3**

Figure 4: relations among dimension, scaling factor, and mass scaling factor for each shape

We could say that what it means for a shape to be, say, 2-dimensional, or what puts the 2 in 2-dimensional, is that when we scale it by some factor, its mass is scaled by that factor raised to the second power. We can express this relation as below:

	1-Dimensional	2-Dimensional	3-Dimensional	D-Dimensional
Length (L)	sL	sL	sL	sL
Mass (M)	$s^1 M$	$s^2 M$	$s^3 M$	$s^D M$

Figure 5: s is the scaling factor and s^D is the mass scaling factor

So if this is our conception of dimension, what should the dimensionality of a Sierpinski Triangle be? We can see that the scaling factor s is $1/2$, its mass goes down by the power of its dimension D. Since it is perfectly self-similar, we know that we want the mass to go down by a factor of $1/3$. So what is the number D such that raising $1/2$ to the power of D gives us $1/3$? i.e.:

$$\left(\frac{1}{2}\right)^D = \left(\frac{1}{3}\right) \Rightarrow 2^D = 3$$

We can obtain the answer for D by using logarithm:

$$\log_2(3) = 1.58496250\ldots$$

Thus, we can say that the Sierpinski triangle is approximately 1.585-dimensional. If you want to describe a Sierpinski triangle's mass, neither length nor area seem like the fitting notion. As shown previously, its length would be infinite and its area would turn out to be zero. Instead, what we want is the 1.585-dimensional analog of its length. It would be more helpful if we look at a few more fractals and calculate their dimension to get a better idea.

4.2.1 Examples

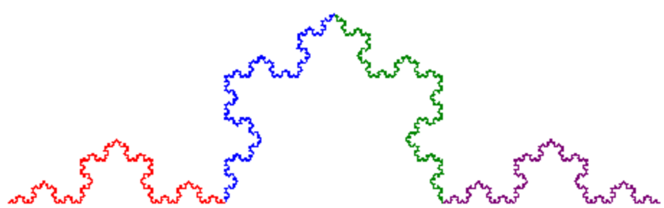

Figure 6: The Koch Curve

As shown in *Figure 6*, we have a Koch Curve which is essentially just a side of the Koch snowflake. We can see that the original curve is consisted of 4 identical copies of itself. As shown in *Figure 1*, the scaling factor is $1/3$ by the construction of the Koch snowflake. To further explain, the Koch snowflake can be constructed by starting with an equilateral triangle, then recursively altering each line segment as follows:

- divide the line segment into three segments of equal length.
- draw an equilateral triangle that has the middle $1/3$ segment from step 1 as its base and points outward.
- remove the line segment that is the base of the triangle from step 2.

The Koch snowflake is the limit approached as the above steps are followed indefinitely. So now we know that the Koch curve has a scaling factor of $1/3$. As shown in *Figure 6*, its mass scaling factor is $1/4$. We can use the same method described above to calculate its dimension D:

$$D = \log_3(4) = 1.26185950\ldots$$

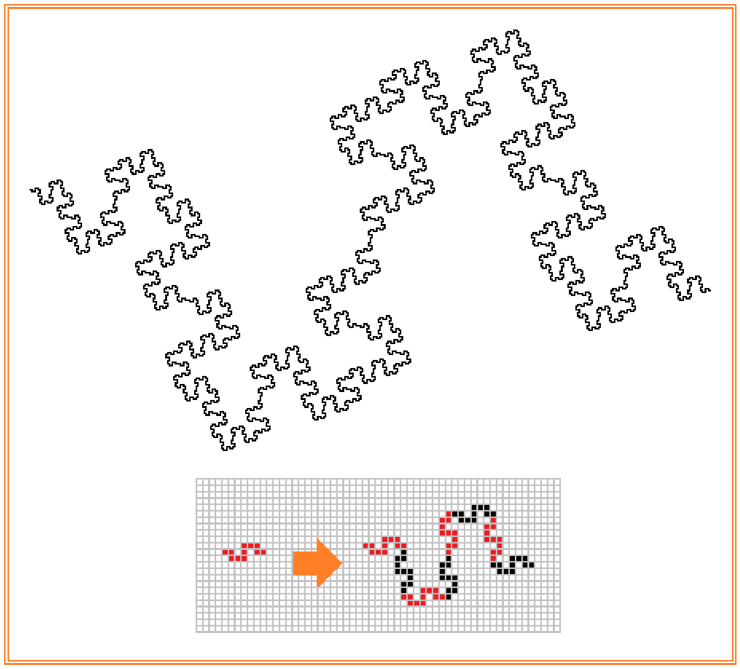

Figure 7: The Minkowski Sausage

We can test our knowledge on another example. The fractal in *Figure 7* is called the Minowski sausage.It is a fractal first proposed by and named for Hermann Minkowski as well as its casual resemblance to a sausage or sausage links. The initiator is a line segment and the generator is a broken line of eight parts one fourth the length.[**] Analogously, we know that its scaling factor is 1/4 and its mass scaling factor is 1/8. Therefore,

$$D = \log_4(8) = 1.5\ldots$$

We can see that the Minkowski sausage is exactly 1.5-dimensional.

4.2.2 More on Fractal Dimension

The examples above was just a toy model for what fractal dimension really is about. Everything we have described so far can be described as "self-similar dimension". It does a good job of making the concept of fractal dimension seem at least somewhat reasonable, but there is a problem : its not a general mathematical notion. When we described how the mass of shape should change, we relied on the self-similarity of the shapes, which allows as to build them up from smaller copies of themselves. But that seems unnecessarily restrictive because most shapes are not at all self-similar. So how do we measure the dimension of a shape that is not self-similar? In order to answer that question, we will have to make this idea of mass scaling factor more mathematically rigorous.

In fractal geometry, the Minkowski–Bouligand dimension, also known as Minkowski dimension or box-counting dimension, is a way of determining the fractal dimension of a set S in a Euclidean space R^n, or more generally in a metric space. To put it into layman's terms, if we put some 2-dimensional fractal shape, say the coastline of Britain (*Figure 8*), on a plane with the grid and highlight all of the grid square that is touching the shape. Now we can count how many there are and call that number N_1. Then we scale the grid down by $1/x$ of its original length, where x is the scaling factor, and count the number of the intersected squares again and call it N_2. The idea is that the fraction of the box counts using the smaller grid over the box counts of the larger grid is approximately the scaling factor to the power of its dimension D:

$$\frac{N_2}{N_1} \approx x^D$$

and if we scale down the grid infinitely by the scaling factor, we will eventually get a number that is very close to its actual dimension.

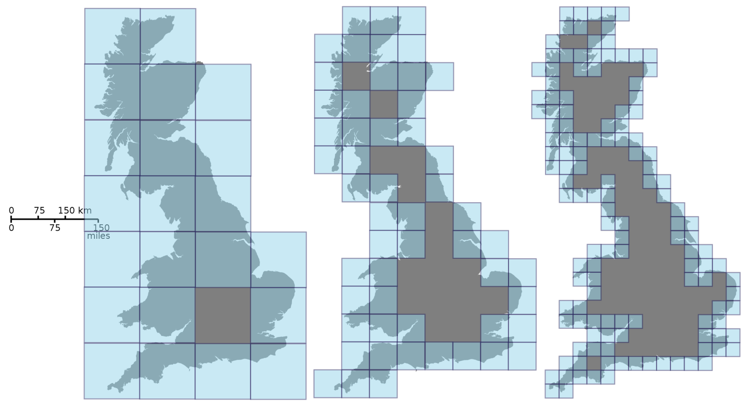

Figure 8: The Minkowski–Bouligand dimension, or box-counting dimension [7]

Moreover, it is worth to mention the Hausdorff dimension is a measure of roughness, or more specifically, fractal dimension, that was first introduced in 1918 by mathematician Felix Hausdorff. It is a successor to the simpler, but usually equivalent, box-counting or Minkowski–Bouligand dimension.[7]

5 Applications of Fractals

Although fractal geometry is quite a difficult concept to grasp, it does not stop people from using fractals for real world applications. In this section, we will talk about the fractals in engineering, electronics, chemistry, etc.

5.1 Fractal Cable

People have been using fractals for centuries in the engineering of super-strong cables. Long before we built the Golden Gate Bridge, the Inca people in South America were building bridges across canyons that were hundreds of feet wide. The cables in these bridges were woven by hand out of long strands of stiff qoya grass. Just as in a modern steel cable, a small number of fibers are woven into a larger fiber, several of which are then woven into a larger rope. This repetitive, fractal pattern provides great strength, and it allows a very long cable woven from thousands of individual pieces of grass no longer than a meter or two each.

> Garcilasco de la Vega, in 1604, reported on the cable-making techniques [of the Inca]. The fibers, he wrote, were braided into ropes of the length necessary for the bridge. Three of these ropes were woven together to make a larger rope, and three of them were again braided to make a still larger rope, and so on. The thick cables were pulled across the river with small ropes and attached to stone abutments on each side."[6]

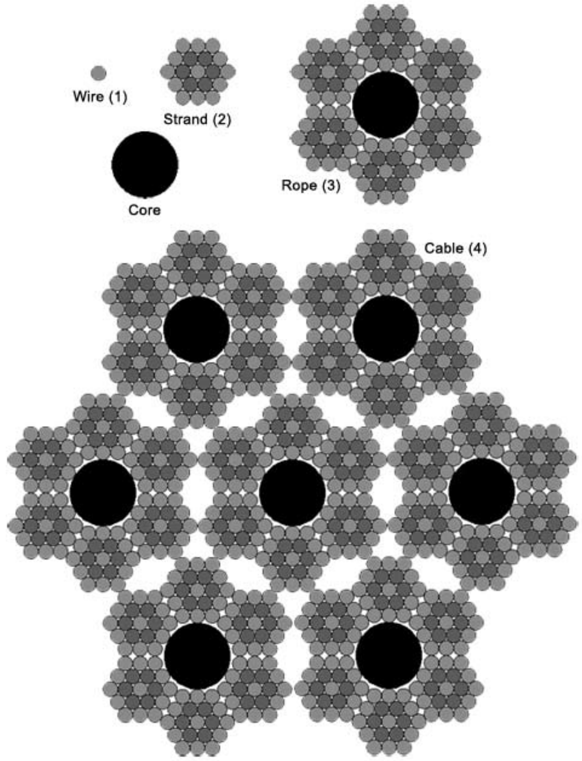

Figure 9: A possible configuration of a steel cable bundle in cross section

Modern engineers still use the same idea in the construction of high strength cables. Cable technology is essential for building suspension bridges. The Inca weavers and the modern steel cable makers both use a repetitive process to create strong, fractal cable patterns. As shown in Figure 9, a steel cable is formed from a bundle of smaller cables which themselves are formed of smaller bundles, etc. It is the fractal cable technology that makes the engineering masterpieces such as the Golden Gate Bridge possible.

5.2 Fractal Antennas

Fractal patterns can be found in commercially available antennas, produced for applications such as cellphones and Wi-Fi systems. Regular antenna has to be cut for one type of signal, and they usually work best when their lengths are certain multiple of their signals' wavelengths. Thus, FM radio antennas can only pick up FM radio stations, TV antennas can only pick up TV channels, etc. However, fractal antennas are different. As the fractal repeats itself more and more, the fractal antenna can pick up more and more signals instead of just one. The self-similar structure of fractal antennas gives them the ability to receive and transmit over a range of frequencies, allowing powerful antennas to be made more compact. Although fractal antennas can receive many different types of signals, they can't always receive each type of signal as well as an antenna that was cut for it.[4]

Figure 10: Fractal antenna

5.3 Fractal Fluid Mixing

The fractals below are developed by Amalgamated Research Inc and are licensed to industrial producers who need to mix fluids together very carefully. The standard way of mixing fluids, stirring, has some important drawbacks. First of all, it produces turbulent mixing, which is unpredictable and uncontrollable, so two different batches may not end up identically mixed. Secondly, turbulent mixing is energy intensive and is disruptive to delicate structure.[4]

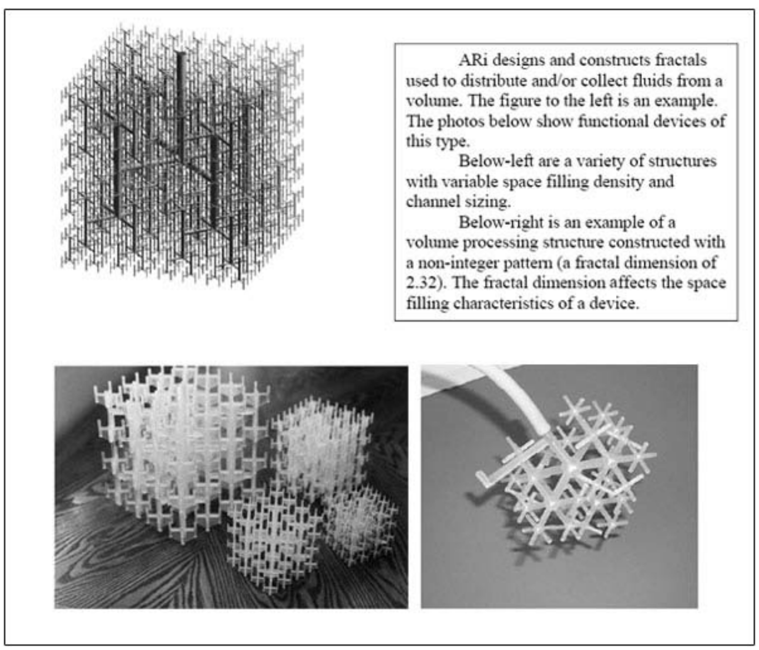

Figure 11: An excerpt from a publication of Amalgamated Research Inc, showing some manufactured fractals used for fluid mixing. Image courtesy of Amalgamated Research Inc.

The engineered fractal fluid mixers provide a different solution to the problem of mixing. The fractal fluid mixer is an actual physical branching fractal network of tubes that can distribute fluid thoroughly into a chamber containing another fluid. Scaling and distribution are common fluid handling requirements in processes that require mixing or geometry transitions. The self-similar and deeply scaled structure of fractals allow them to directly and efficiently address uniformity of flow and process scale-up.[ARI***]This system provides a low-energy reproducible way of mixing chemicals. Fractal networks with different space-filling properties can be custom-made for chemical systems requiring different properties.

Examples of fluid-mixers like these have been used in fields as diverse as high-precision epoxies, the manufacturing of sugar, and also used to sample and determine small quantities of biological molecules.[1]

6 Conclusions and Future Research

Fractal geometry is certainly a complex and yet fascinating topic in mathematics. It completely changed the way we think about geometry. Fractional geometry allows us to capture the roughness of irregular shapes and helps us to better understand the complexity of nature.

Some of my peers provided me some possible future research topics. One of the topics involve fractal patterns in music composition. The concept of "Shepard Tone" that demonstrates a fractal pattern in rhythm was mentioned by one of my peers. Another suggestion was to further investigate the fractal pattern in policy making. A lot of policies are made base on some very fundamental ideas. However, depending on factors like economy, politics, education, etc., a simple idea can diverge in many different ways, like a snowflake. That is why we have countries with completely different policies and laws. Lastly, I am personally very interested in doing research on the applications of fractals in computer science, more specifically, computer graphics. As I mentioned in the topic selection section, what motivated me to talk about fractals is actually the movie "Doctor Strange". In fact, many visual-effects artists have been using fractals in sci-fi movies to illustrate concepts like parallel universe, other dimensions, or intergalactic traveling. The advantage of using fractal to create visual arts is that it only takes a simple math formula or statement to generate some infinitely complex objects, which can be very cost-efficient for filmmakers.

References

[1] Amalgmated Research Inc, Fractal fluid control, hhttps://www.arifractal.net/technologies/fractal-fluid-control/

[2] Benoit Mandelbrot, *The Fractal Geometry of Nature*, page.1-2

[3] Craig Haggit, How Fractals Work, https://science.howstuffworks.com/math-concepts/fractals.htm#:~:text=Advertisement-,Before%20They%20Were%20Fractals,Off%20Kanagawa%22%20the%20early%201800s.&text=When%20most%20people%20think%20about,them%20all%2C%20the%20Mandelbrot%20Set., accessed October 10, 2020.

[4] Fractal Foundations, Fractal Application, http://fractalfoundation.org/OFC/OFC-12-2.html, accessed October 10, 2020.

[5] Internal, What is a Fractal, https://iternal.us/what-is-a-fractal/, accessed October 10, 2020.

[6] John Noble Wilford, How the Inca Leapt Canyons, https://www.nytimes.com/2007/05/08/science/08bridg.html?pagewanted=1&_r=1, accessed October 10, 2020.

[7] Minkowski–Bouligand dimension, Wikipedia, Wikimedia Foundation https://en.wikipedia.org/wiki/Minkowski%E2%80%93Bouligand_dimension, accessed October 10, 2020.