The Mathematics of Music Composition

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1 Introduction

Music is inherently mathematical, rooted in the manipulation of patterns and space between values. Although the way we decide to arrange these patterns and what we deem as "pleasing" or "meaningful" varies largely between cultures and even individual preferences, examining music through the lense of mathematics provides a deeper insight into how we can concretely describe something as abstract as art. This paper serves as an attempt to break down some of the core tenets of traditional Western music theory in a way that is accessible to even those without a background in music composition.

2 Background and Basic Musical Terminology

In order to have a discussion about the mathematical nature of musical conventions, it is important to acknowledge the context in which these conventions are accepted. Western music theory is by no means representative of all facets of music. For example, a lot of traditional East Asian music relies heavily on the pentatonic scale, an arrangement of semitones which is less commonly utilized in Western music. The pentatonic scale is also commonly used in tuning the Mbira, an instrument often used in Zimbabwean music, which in itself has polyrhythms and structures not utilized in Western theory. In fact, all of the rules established in Western music theory are often broken, even within Western composition. With these caveats in mind, from this point forward we will be discussing music composition in the limited context of traditional Western theoretical conventions.

2.1 Defining Western Music Theory

When we think of Western music theory, we are often considering music in what is known as the "Tonal Era", beginning in 1600 with the genesis of the Baroque Period and ending in 1900 with the conclusion of the Romantic Period; (in between these two is the Classical Period). Western music theory is centered around the octave, an interval of music that is divided into twelve equal parts, called semitones. With this basis, a number of conventions arose for arranging these semitones into patterns that would sound "pleasing" to the ear.

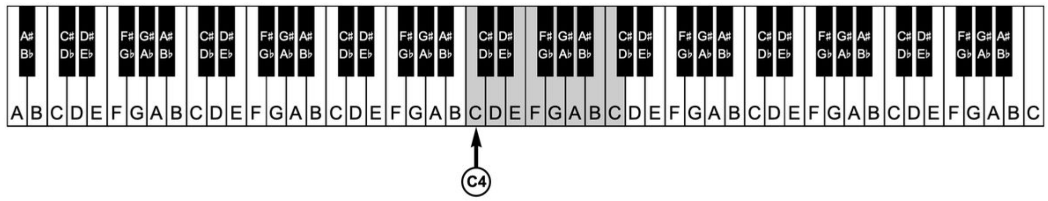

Figure 1: Semitone Representation on a Keyboard

2.2 Important Musical Terminology

There are a few important terms which we will define before translating into more mathematical notation. We often use a piano keyboard to help visualize the relationship between different musical conventions (see Figure 1). A *semitone* is the smallest value in Western music. The space between each successive pitch on the keyboard is a semitone. Arranging twelve pitches separated by semitones in ascending order gives us an *octave*. In Figure 1, the gray area represents one octave. An *interval* is the number of semitones (or the "space") between two pitches. Different arrangements of these semitones establish different *keys* and *qualities*, which are the foundation for what pitches we consider when writing music and how the music will sound after it is written (happy, sad, energetic, etc.). Finally, a *chord* is a series of pitches sounded at the same time, and the order in which chords are played is called the *chord progression*.

3 Translating Musical Conventions to Mathematical Notation

Having established a framework in which we are viewing music (through Western theoretical conventions), and with a basic understanding of some musical terminology, we are now able to re-imagine musical concepts as mathematical formulas.

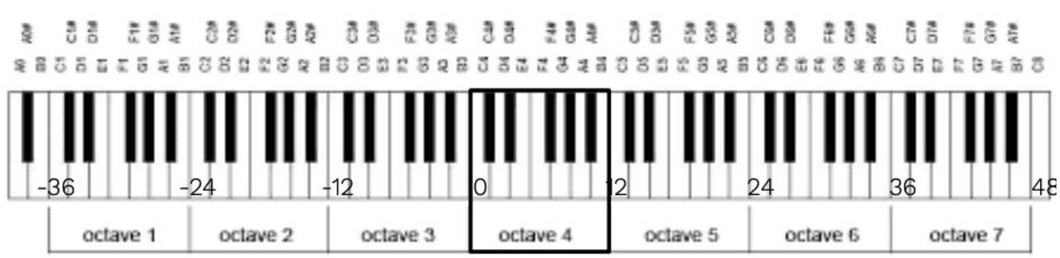

Figure 2: Reimagining Pitches as Numerical Values

3.1 Pitches as a Set

Every octave contains 12 distinct pitches separated by semitones, arranged in a particular order. Because every octave contains the same 12 pitches in the same order, the only distinction between octaves is how high or low the frequency of these pitches is. For our purposes, we will consider each octave a set of 12 elements, as follows:

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$$

Looking at Figure 2, we can see how each of these elements relates back to the pitches. The pitch labeled "0" on the keyboard in Figure 2 is what is called "Middle C" in music, and it will be the starting point for labeling the remaining pitches numerically. After 12 distinct pitches, we enter a new octave, which has the same pitches in the same order, just played at a different frequency. Because of this, we will be considering each element in our set of pitches as equivalent modulo 12.

Another way to think of this is mapping each of the 12 distinct pitches to a number 0-11, as follows in Figure 3.

C ↦ 0	C#/Db ↦ 1	D ↦ 2	D#/Eb ↦ 3	E ↦ 4	F ↦ 5
F#/Gb ↦ 6	G ↦ 7	G#/Ab ↦ 8	A ↦ 9	A#/Bb ↦ 10	B ↦ 11

Figure 3: Mapping Pitches to Distinct Numerical Elements

3.2 Computing Intervals

Now that we have a method of describing pitches numerically, we can begin to compute intervals. Figure 4 describes each interval as it relates to a number of semitones. The musical name of each of these intervals is, for our purposes, less important than the actual value associated with each interval.

Interval Name	**# of Semitones**	**Interval Name**	**# of Semitones**	**Interval Name**	**# of Semitones**	**Interval Name**	**# of Semitones**
minor 2	1	Major 3	4	Perfect 5	7	minor 7	10
Major 2	2	Perfect 4	5	minor 6	8	Major 7	11
minor 3	3	Tritone	6	Major 6	9	Perfect 8	12

Figure 4: Values of the Intervals

With this information, we can easily compute the value of a new pitch at a desired interval above or below a given pitch, as follows:

Value at an Interval Above a Given Pitch:

[(desired pitch from our set) + (# of semitones of desired interval)] (mod12) ≡ value of new pitch

Value at an Interval Below a Given Pitch:

[(desired pitch from our set) - (# of semitones of desired interval)] (mod12) ≡ value of new pitch

For example, if we wanted to find what pitch occurs at a minor 7 above F#, we would do the following computation:

$$(6+10)\ (\text{mod}12) = 16\ (\text{mod}12) \equiv 4\ (E)$$

Using this method, we are able to find pitches at any desired interval away from a given pitch.

4 The Composition Process

With all of these tools, we are now able to begin composing music. There are three major steps we want to consider when writing music: Establishing a key and quality, choosing a chord progression, and writing a melody.

4.1 Establishing a Key and Quality

Western music theory is often centered around one of two qualities of keys: a Major key, which is often associated with happiness and energy, or a minor key, which is often associated with sadness and slowness. For our purposes, we are going to think of Major and minor keys as a matrices created through the following operations:

Major Keys

$$[0\ 2\ 4\ 5\ 7\ 9\ 11] + k[1\ 1\ 1\ 1\ 1\ 1\ 1] = [a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7]$$

minor keys

$$[0\ 2\ 3\ 5\ 7\ 8\ 10] + k[1\ 1\ 1\ 1\ 1\ 1\ 1] = [a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7]$$

Here, the first matrix consists of intervals above the pitch we have chosen as the basis of our key. The second matrix is multiplied by the scalar, k, where k is the pitch we want to represent the basis of our key. This creates a new 1x7 matrix, which we are going to from now on refer to as the key.

For example, if we wanted to find the key corresponding to the value 7 with a major quality, we would do so as follows:

$$[0\ 2\ 4\ 5\ 7\ 9\ 11] + 7[1\ 1\ 1\ 1\ 1\ 1\ 1] = [7\ 9\ 11\ 12\ 14\ 16\ 18]$$
$$[7\ 9\ 11\ 12\ 14\ 16\ 18] \equiv [7\ 9\ 11\ 0\ 2\ 4\ 6] \pmod{12}$$

4.2 Choosing a Chord Progression

Just as we described a key as a matrix, we are now going to describe chords as matrices. We are going to examine three qualities of chords: Major, minor, and diminished. These chord qualities are established by the following formulas:

Major Chords

$[0\ 4\ 7\ 0] + a_i[1\ 1\ 1\ 1]$

minor Chords

$[0\ 3\ 7\ 0] + a_i[1\ 1\ 1\ 1]$

diminished Chords

$[0\ 3\ 6\ 0] + a_i[1\ 1\ 1\ 1]$

Here, the first matrix consists of intervals above the pitch we have chosen for the basis of our chord. The second matrix is multiplied by the scalar, a_i, where a_i is the element of our key that we want to represent the basis of our chord. This creates a new 1x4 matrix, which we are from now on going to refer to as a chord.

There are a number of pre-established chord progressions we can choose from to compose a piece of music. Depending on which chord we begin with, we have certain options for which chord we can use next, and so on until we have selected all of the chords we are interested in using. A general convention is that pieces will begin with a “I” chord and end with a “V” chord followed by a “I” chord.

Figure 5 describes the chord progressions for Major and minor keys, as well as how to find each individual chord chord. Again, for our purposes, the musical names for each of these chords are less important than the formulas used to create them.

Chord Progression Conventions in Major Keys

Abrev	Name	Matrix	Goes to
I	Major 1	[0 4 7 0]+a_1[1 1 1 1]	anywhere
ii	minor 2	[0 3 7 0] +a_2[1 1 1 1]	V, vii°
iii	minor 3	[0 3 7 0] +a_3[1 1 1 1]	IV, vi
IV	Major 4	[0 4 7 0]+a_4[1 1 1 1]	V, I, vii°
V	Major 5	[0 4 7 0]+a_5[1 1 1 1]	I, vi
vi	minor 6	[0 3 7 0] +a_6[1 1 1 1]	ii, IV, V
vii°	diminished 7	[0 3 6 0] +a_7[1 1 1 1]	I, V

Chord Progression Conventions in minor Keys			
Abrev	**Name**	**Matrix**	**Goes to**
i	minor 1	[0 3 7 0] +a_1[1 1 1 1]	anywhere
ii°	diminished 2	[0 3 6 0]+a_2[1 1 1 1]	V, VI
III	Major 3	[0 4 7 0]+a_3[1 1 1 1]	iv, VI
iv	minor 4	[0 3 7 0] +a_4[1 1 1 1]	V, i, vii°
V	Major 5	[0 4 7 0]+a_5[1 1 1 1]	i, VI
VI	Major 6	[0 4 7 0]+a_6[1 1 1 1]	ii°, iv, V
vii°	diminished 7	[0 3 6 0] +a_7[1 1 1 1]	i, V

Figure 5: Chord Progression Conventions

Using these conventions, we can create whatever chord progressions we desire. For example, using the matrix for the “7 Major Key” that we established earlier, we can now create this simple chord progression:

I -> IV -> V -> I

$[0\ 4\ 7\ 0]+7[1\ 1\ 1\ 1] = [7\ 11\ 14\ 7] \equiv [7\ 11\ 2\ 7]\ (\text{mod}12)$
$[0\ 4\ 7\ 0]+0[1\ 1\ 1\ 1] = [0\ 4\ 7\ 0] \equiv [0\ 4\ 7\ 0]\ (\text{mod}12)$
$[0\ 4\ 7\ 0]+2[1\ 1\ 1\ 1] = [2\ 6\ 9\ 2] \equiv [2\ 6\ 9\ 2]\ (\text{mod}12)$
$[0\ 4\ 7\ 0]+7[1\ 1\ 1\ 1] = [7\ 11\ 14\ 7] \equiv [7\ 11\ 2\ 7]\ (\text{mod}12)$

4.3 Writing a Melody

Having established a chord progression, writing a melody using these conventions becomes relatively simple. Wherever we decide to use a chord, we can compose a melody using exclusively pitches from that chord until the next chord is used. If we do this, we know that the melody we write will sound consonant (pleasing to the ear), regardless of the order of pitches or how many of them are played at the same time. As this part of the composition process is entirely subjective, there are not specific formulas associated with melody composition.

4.4 An Example of this Method (Pachelbel's Canon in D)

Figure 6 provides a concrete example of this methodology being put to use. The key here is related to the pitch "2" in our set of pitches, and is Major in quality. The matrix given by this key is [2 4 6 7 9 11 1]. The chord progression here is I -> V -> vi -> iii -> IV -> I -> IV -> V. The matrices used in the chord progression are as follows: I: [2 6 9 2], V: [9 1 4 9], vi: [11 2 6 11], iii: [6 9 1 6], IV: [7 11 2 7], I: [2 6 9 2], IV: [7 11 2 7], V: [9 1 4 9]. The matrices of each chord that appears have been labeled above the associated chord, and the values of the pitches that are being used have been labeled below the associated pitches.

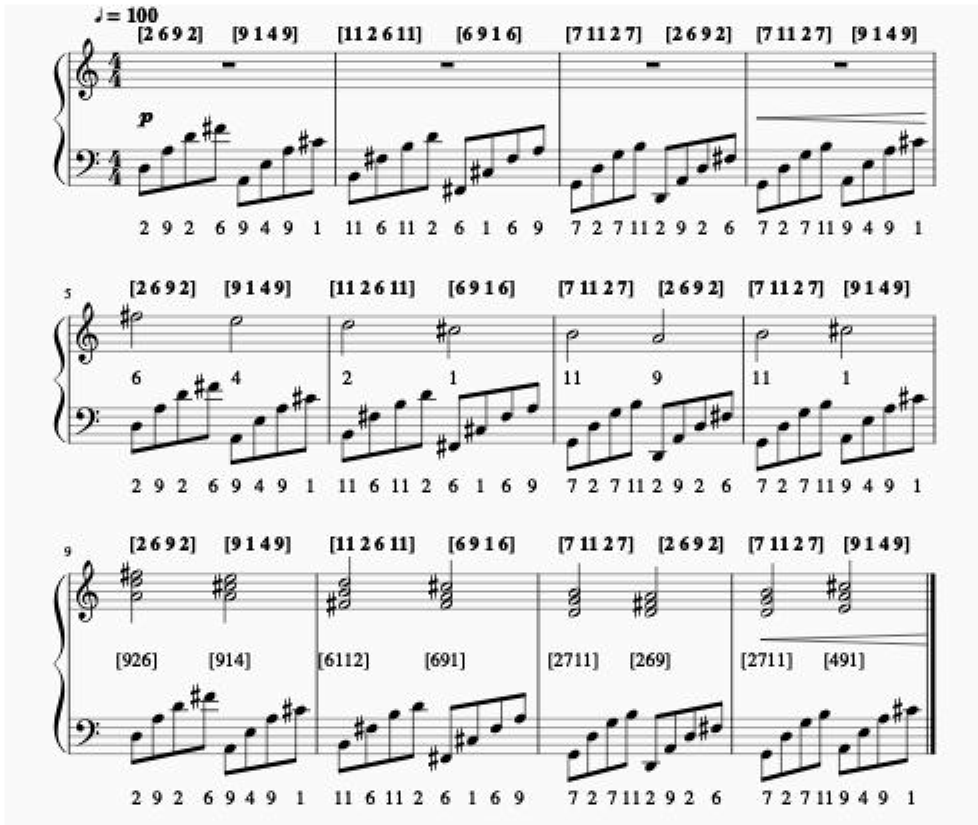

Figure 6: Pachelbel's Canon in D

5 Conclusions

The goal of this presentation was to make music composition and theory seem accessible to an audience that may not have much of a background or interest in music. While describing music theory in this way may seem abstract to some who are removed from the field, these methodologies are in fact anything but trivial. Skilled musicians and composers think of music as relationships between pitches and understand how to manipulate those pitches to create pleasing sounds; while they may not always break down music to the formulaic degree of that presented in this paper, the general process is the same. Looking at music in this way is also valuable because it begins to dismantle the barriers between mathematics and art. We often view the two as separate entities, when in fact they are fundamentally entangled. This connection illustrates the importance of teaching the arts in addition to core education requirements; while mathematics is useful in its own right, when explored through the lense of music, its applied beauty becomes apparent.

References

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