The Mathematics of Twelve Tone Music

Bethany Shears

1 Introduction

Throughout history, musical conventions have been manipulated, broken, and rewritten to reflect the changing values of the eras in which they are established. Around the turn of the 20th century, Western music went through such a period of musical experimentation. A push towards stretching the boundaries of what could be considered "music" and how it could be produced lead to the development of revolutionary atonal music strategies. This paper is a companion piece to an earlier paper, "The Mathematics of Music Composition," and for that reason assumes that some basic understanding of musical terminologies and their translation to mathematical notation is inherent. It serves as an attempt to explore twelve tone music composition through a mathematical lens, and to break down its key concepts in a way that is accessible to even those without a background in music.

2 Recapitulation and Background

Despite assuming a certain fluency in the concepts described by the preceding paper, there is some terminology that is particularly useful for this discussion. Hence, the following provides an overview of some fundamental concepts without expanding upon them in great detail, before introducing the main subject material. For a more thorough understanding of these conventions, please refer to "The Mathematics of Music Composition".

2.1 Recapitulation of Necessary Terminology

There is some important musical terminology to review before discussing twelve tone music in detail. A *semitone* is the smallest value in Western music. The space between each successive pitch on the keyboard is a semitone. An *interval* is the number of semitones (or the "space") between two pitches. There is an assumption in Western music that there are only twelve distinct pitches, which can be played at higher or lower frequencies. For our purposes, we will consider these twelve distinct pitches a set of 12 elements, as follows:

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$$

Each of these elements is to be considered modulo 12. Another way to think of this is mapping each of the 12 distinct pitches to a number 0-11, as follows in Figure 1.

C ↦ 0	C#/Db ↦ 1	D ↦ 2	D#/Eb ↦ 3	E ↦ 4	F ↦ 5
F#/Gb ↦ 6	G ↦ 7	G#/Ab ↦ 8	A ↦ 9	A#/Bb ↦ 10	B ↦ 11

Figure 1: Mapping Pitches to Distinct Numerical Elements

Finally, it is important to recall that to find an interval above a given pitch, we add the number of semitones associated with the interval to the number associated with the pitch, and reduce modulo 12. (Similarly, to find an interval below a given pitch, we instead subtract the number of semitones associated with the interval). Figure 2 details the intervals as they relate to a number of semitones.

Interval Name	# of Semitones	Interval Name	# of Semitones	Interval Name	# of Semitones	Interval Name	# of Semitones
minor 2	1	Major 3	4	Perfect 5	7	minor 7	10
Major 2	2	Perfect 4	5	minor 6	8	Major 7	11
minor 3	3	Tritone	6	Major 6	9	Perfect 8	12

Figure 2: Values of the Intervals

2.2 A Brief History of Twelve Tone Music

Traditional Western music theory conventions are used to describe tonal music, or music that is centered around a certain key. These practices are rooted in a dedication to create constant music that is pleasing to the ear. Around the turn of the 20th century, however, there was a push towards more experimental and dissonant music composition, inspired by the increasingly chromatic pieces written by Romantic Era composers such as Franz Liszt.

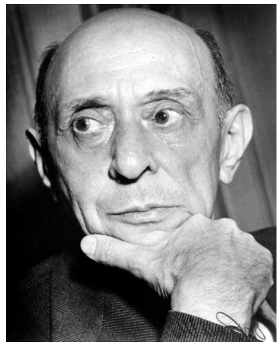

Figure 3: Photograph of Arnold Schoenberg (1874-1951)

Austrain composer Arnold Schoenberg (Figure 3) was inspired by the increasingly dissonant pieces of the Late Romantic period to experiment with methods of creating music that, instead of relying on the ordering of pitches within a certain key, gave equal importance to each one of the twelve distinct pitches throughout a composition. This musical philosophy eventually gave rise to the establishment of the twelve tone technique, a formulaic approach to music composition that attempts to give each pitch equal weight, and the often-cited beginning of the serialist movement in post-tonal musical composition.

3 Translating Twelve Tone Composition to Mathematics

Twelve tone technique is different from traditional Western techniques not only in its rejection of a tonal center, but also in the relative rigidity of the structure it creates. Whereas traditional Western music has a lot of freedom in the ordering of pitches and the ways they can be manipulated, twelve tone compositions are limited to a certain pattern and usage of pitches. This arises from a series of transformations upon an original pattern of pitches, and follows quite formulaically.

3.1 The Tone Row

The aspect of twelve tone compositions that arguably provides the most originality to a piece comes from the creation of an original Tone Row. The Tone Row, (or "Note Row," in Europe), is an ordering of all twelve distinct pitches in any way that is desired by the composer. Each pitch can be included only once. Generally, the more arbitrary the ordering of pitches within the Tone Row is, the more atonal and eclectic the produced composition will sound.

0	9	7	3	4	5	2	11	10	8	1	6

Figure 4: Example Tone Row

Figure 4 uses a 12-element array to represent an example Tone Row using the numerical notation for pitches that was previously established. This original ordering of the Tone Row is referred to as "Prime Form," and is often denoted by P.

3.2 Manipulation of the Tone Row

The Tone Row now serves as the basis for creating all possible patterns of pitches that can be utilized in a composition. There are three basic operations that can be performed upon the Prime Form (P) of a Tone Row to create said patterns, namely: Retrograde (denoted by R), Inversion (denoted by I), and Retrograde Inversion (RI).

P	0	9	7	3	4	5	2	11	10	8	1	6
R	6	1	8	10	11	2	5	4	3	7	9	0

Figure 5: Retrograde Compared to Prime Form

The first operation that can be performed upon the Prime Form of the Tone Row is the Retrograde. The Retrograde takes the order of elements of the Prime Form array and flips it to create a new ordering of the Tone Row, as exemplified in Figure 5.

P	0	9	7	3	4	5	2	11	10	8	1	6
I	0	3	5	9	8	7	10	1	2	4	11	6

Figure 6: Inversion Compared to Prime Form

The next operation that can be performed upon the Prime Form of the Tone Row is the Inversion. The Inversion takes the elements of the Prime Form array and functionally inverts the intervals between each pitch, as exemplified in Figure 6. Recalling that finding an interval above or below a given pitch requires adding or subtracting a number of semitones, it becomes obvious that the Inversion of the Prime Form will require some formula to manipulate the original pattern of pitches. The formula to create an Inversion is as follows:

P_1: Stays the same
P_i: $[12 - (P_i - P_1) + P_1] \pmod{12}$

Here, the subscript associated with P refers to the position of the element in the array, with P_1 being the first element in the array and i being a natural number such that $1 < i < 13$.

P	0	9	7	3	4	5	2	11	10	8	1	6
I	0	3	5	9	8	7	10	1	2	4	11	6
RI	6	11	4	2	1	10	7	8	9	5	3	0

Figure 7: Retrograde Inversion Compared to Prime Form and Inversion

The final operation that can be performed upon the Prime Form of the Tone Row is the Retrograde Inversion. The Retrograde Inversion takes the order of elements of the Inversion array and flips it to create a new ordering of the Tone Row, as exemplified in Figure 7.

3.3 The Twelve Tone Matrix

The real significance of the Tone Row is that, in creating an ordering of all twelve possible pitches, it establishes a pattern of intervals between said pitches. Therefore, it is possible to recreate this pattern of intervals starting with any of the twelve distinct pitches. Essentially, this is transposing the pattern to begin on a different pitch. In order to create and keep track of these new iterations of the Tone Row, a Twelve Tone Matrix is utilized. The term “matrix,” in this sense, may be somewhat of an abuse of the term; in reality, a 12x12 grid is created and filled with the possible arrays representing the Tone Rows. However, this process is not arbitrary, and there are a number of rules to follow when establishing a Twelve Tone Matrix.

0	9	7	3	4	5	2	11	10	8	1	6
3											
5											
9											
8											
7											
10											
1											
2											
4											
11											
6											

Figure 8: The Beginnings of a Twelve Tone Matrix

After creating a blank 12x12 grid, the first step in creating a Twelve Tone Matrix is to fill in the first row with the Prime Form array of the original Tone Row. Then, the first column can be filled in with the Inversion of the Prime Form (see Figure 8). These two arrays create a basis for filling in the remaining entries in the grid. Each remaining row is then filled in according to the following formula:

P_{a1}: Stays the same

P_{ai}: $[(P_{a1} - P_{11}) + P_{1i}]$ (mod12)

Here, the subscript associated with P refers to the position of the element in the grid, with P_{11} being the first element in the first row, P_{a1} being the first element in the ath row, and P_{1i} being the ith element in the 1st row. Then, both a and i are natural numbers such that $1 < i < 13$ and $1 < a < 13$. Once this process has been completed for each row, the Twelve Tone Matrix is complete (see Figure 9).

0	9	7	3	4	5	2	11	10	8	1	6
3	0	10	6	7	8	5	2	1	11	4	9
5	2	0	8	9	10	7	4	3	1	6	11
9	6	4	0	1	2	11	8	7	5	10	3
8	5	3	11	0	1	10	7	6	4	9	2
7	4	2	10	11	0	9	6	5	3	8	1
10	7	5	1	2	3	0	9	8	6	11	4
1	10	8	4	5	6	3	0	11	9	2	7
2	11	9	5	6	7	4	1	0	10	3	8
4	1	11	7	8	9	6	3	2	0	5	10
11	8	6	2	3	4	1	10	9	7	0	5
6	3	1	9	10	11	8	5	4	2	7	0

Figure 9: Completed Twelve Tone Matrix

Notice that each pitch is repeated only once in every row and column of the Twelve Tone Matrix. This speaks to the purpose of creating music where each pitch receives equal weight in a composition.

-------> Prime Form -------------------------------->

------> Inverse ------>

0	9	7	3	4	5	2	11	10	8	1	6
3	0	10	6	7	8	5	2	1	11	4	9
5	2	0	8	9	10	7	4	3	1	6	11
9	6	4	0	1	2	11	8	7	5	10	3
8	5	3	11	0	1	10	7	6	4	9	2
7	4	2	10	11	0	9	6	5	3	8	1
10	7	5	1	2	3	0	9	8	6	11	4
1	10	8	4	5	6	3	0	11	9	2	7
2	11	9	5	6	7	4	1	0	10	3	8
4	1	11	7	8	9	6	3	2	0	5	10
11	8	6	2	3	4	1	10	9	7	0	5
6	3	1	9	10	11	8	5	4	2	7	0

<------ Retrograde Inversion <------

<------- Retrograde <--------------------------------

Figure 10: Utilizing the Twelve Tone Matrix

This configuration is also useful because it reveals the Prime Form, Retrograde, Inversion, and Retrograde Inversion of every possible iteration of the Tone Row (see Figure 10). Reading the rows left to right gives the Prime Form, while right to left gives the Retrograde. Reading the columns top to bottom give the Inversion, while bottom to top gives the Retrograde Inversion.

3.4 The Rules for Composition

Having created a Tone Row and utilized it to establish a Twelve Tone Matrix, all the tools for composing a piece of music according to the twelve tone technique are now in place. There are a few rules for composing in this style, which, when conformed to, make the composition process follow naturally. First, select any row (or column) in the Twelve Tone Matrix, in any format (Prime Form, Retrograde, Inversion, or Retrograde Inversion). Once a row has begun to be used in a composition, it must be followed to completion. All of the pitches must be played in order, and none may be repeated or skipped. Notes may be played in any octave and for any duration. Pitches may also be played simultaneously, as long as they occur sequentially within the row. Finally, any number of rows (or columns) may be played concurrently, and can have varying lengths and formats. Following these rules exactly to the conclusion of a composition will create a piece of music in conformation with the twelve tone technique.

4 Why This Technique Makes Mathematical Sense

In his essay "Mathematics and the Twelve Tone System: Past, Present, and Future," American composer and music theorist Robert Morris beautifully explained the mathematical intentions of Schoenberg in creating twelve tone music:

> "Schoenberg's phrase, 'The unity of musical space,' while subject to many interpretations, suggests that he was well aware of the symmetries of the system ... he understood that there was a singular two-dimensional 'space' in which his music lived … Indeed, the basic transformations of the row, retrograde and inversion, plus retrograde-inversion for closure (and P as the identity) were ... shown to form a Klein four-group."

As Morris implies, the Prime Form, Retrograde, Inversion, and Retrograde Inversion operations on the Tone Row have been proven to conform to the conventions of the Klein four-group, as is demonstrated in Figure 11.

P	R	I	RI
R	P	RI	I
I	RI	P	R
RI	I	R	P

Figure 11: Association with the Klein Four-Group

Unsurprisingly, the properties of the Klein four-group seem to lend themselves to creating a system for formulating music in which every pitch gets equal weight in composition. Every element of this group is its own inverse, and the product of two (non-identity) elements produces the remaining non-identity element. Additionally, the Klein four-group can be visualized by performing transformations on a rectangle. Just like the symmetries of a rectangle, the Prime Form (P) can be transformed using reflection (left-right flip), inversion (up-down flip), or a combination of these (rotation through 180°). These symmetries can be observed when examining Prime Form, Retrograde, Inversion, and Retrograde Inversion forms of the Tone Row in their standard musical notation (see Figure 12).

Figure 12: Visualization of the Tone Row Operations

5 Conclusions

The goal of this paper was to make twelve tone music composition seem more accessible to those without a background or interest in music, and to examine the mathematical connections within twelve tone technique. It was preceded by a presentation of the aforementioned material, and resulted in a few notable addendums.

5.1 Addendums

Following the presentation given in preparation for this work, an excellent question was raised about whether or not twelve tone music is truly atonal. The majority of the time, if a Tone Row has truly been chosen arbitrarily (as was, arguably, the intention of Schoenberg), the resulting piece is atonal. However, because a Tone Row establishes a pattern of intervals between notes, it could theoretically be manipulated in such a way that the resulting intervals correspond to the intervals inherent within chords and keys associated with traditional Western music theory. For that reason, while twelve tone music is intended to be atonal, it is technically possible to create twelve tone music that gives an illusion of tonality. Still, the method by which it is produced and the purpose of the composition makes twelve tone music distinct from traditional Western compositions.

5.2 Conclusions

Just as examining traditional Western music theory in conjunction with mathematics begins to dismantle the barriers between these two fields, twelve tone technique brings mathematics even further into the musical realm. Perhaps an even more direct application of mathematics than traditional music theories, twelve tone music and serialism question whether an increased formulization of the musical process can create sounds that are still considered music. The further implications of this process beg questions about whether art can be mechanized, or if artificial intelligence can be used to create art. Mathematics and music are so fundamentally entwined that the connections between the two are limitless; it is upon closely examining and expanding their similarities that more complex and beautiful art can be created.

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