# Tessellations: The Art of Space Filling

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Presentation Outline

- Definitions and classifications
- Wallpaper groups
- Connections with fractals
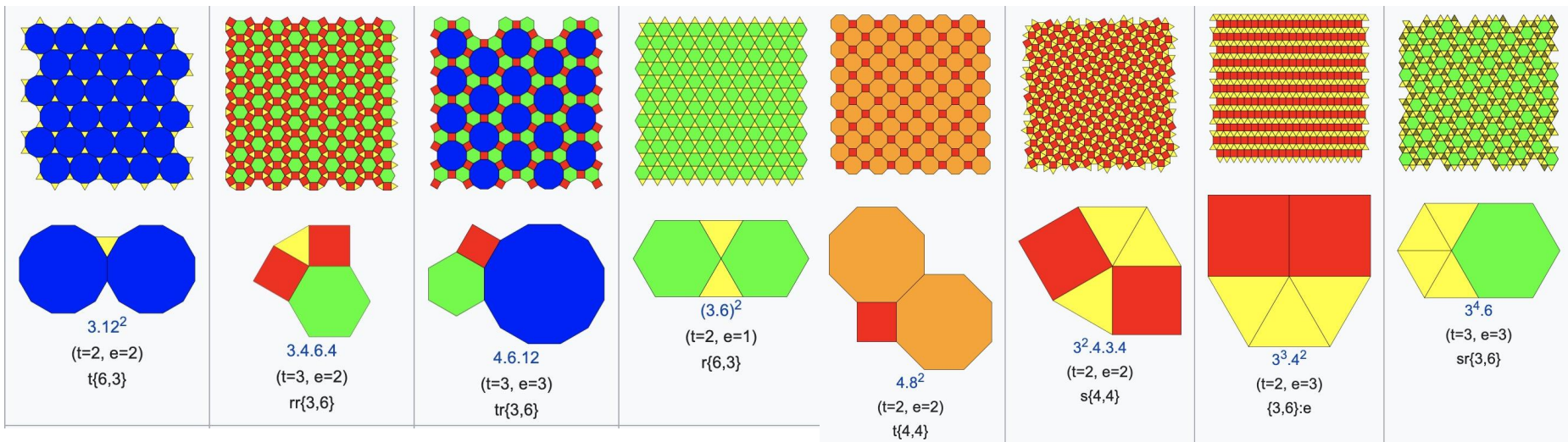- Connections with the Golden Ratio

What is a tessellation?

- A tessellation is a partition of an infinite space into pieces having a finite number of distinct shapes.
- These geometric patterns are called tiles.
- Three types based on the shapes of tiles:
  - **Regular tessellation:** made up of regular polygons of the same size and position

3^6 6^3 4^4
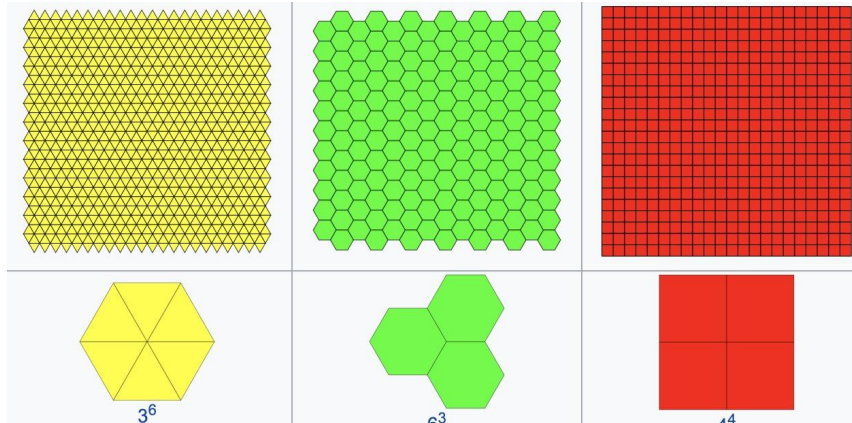
What is a tessellation?

- Three types based on the shapes of tiles:
  - **Semi-regular (Archimedean) tessellation**: made up of more than one type of regular polygons; tiles fit together in an isogonal arrangement

3.12^2
(t=2, e=2)
t{6,3}

$3.4.6.4$
(t=3, e=2)
rr{3,6}

$4.6.12$
(t=3, e=3)
tr{3,6}

$(3.6)^2$
(t=2, e=1)
r{6,3}

4.8^2
(t=2, e=2)
t{4,4}

$3^2.4.3.4$
(t=2, e=2)
s{4,4}

$3^3.4^2$
(t=2, e=3)
{3,6}:e

$3^4.6$
(t=3, e=3)
sr{3,6}

What is a tessellation?

- Three types based on the shapes of tiles:
  - **Irregular tessellation:** no restrictions; could be made up of any kind of geometric shapes
- The Voronoi diagram is an example.

Isometries and Wallpaper Groups

- Four types of symmetries in a plane (isometries of the Euclidean plane)
  - **Translations:** move every point of a tile by the same distance in a given direction
  - **Rotations:** move a tile around a fixed point
  - **Reflections:** mirror a tile
  - **Glide Reflections:** reflection and translation

Isometries and Wallpaper Groups

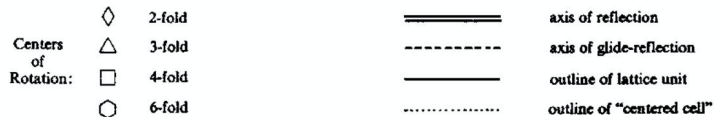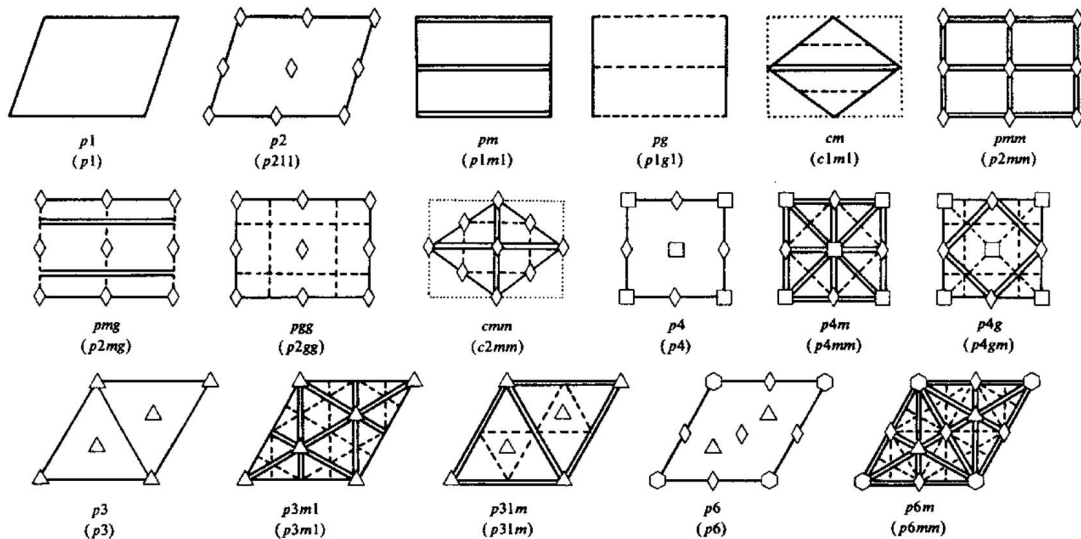- Categorize all repetitive patterns based on symmetries:

  17 Wallpaper Groups

- Formal definition: a type of topologically discrete group of isometries of the Euclidean plane that contains two linearly independent translations.
- 5 kinds of lattices

  Square, Rhombus, Parallelogram, Rectangle, Hexagon
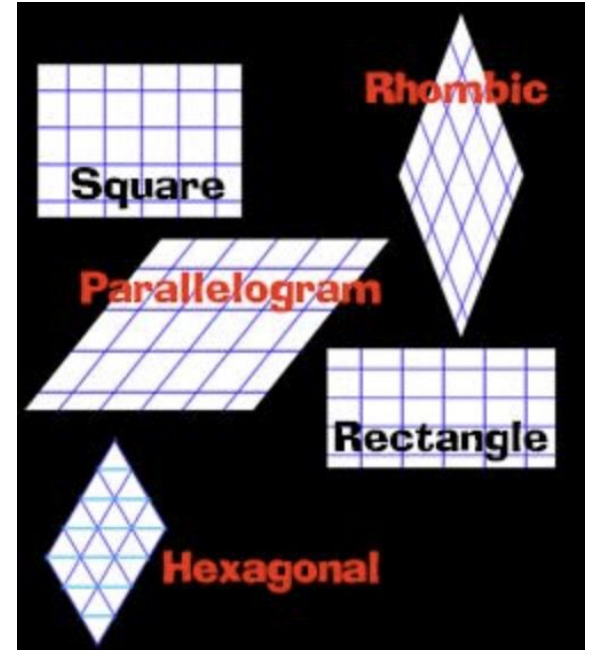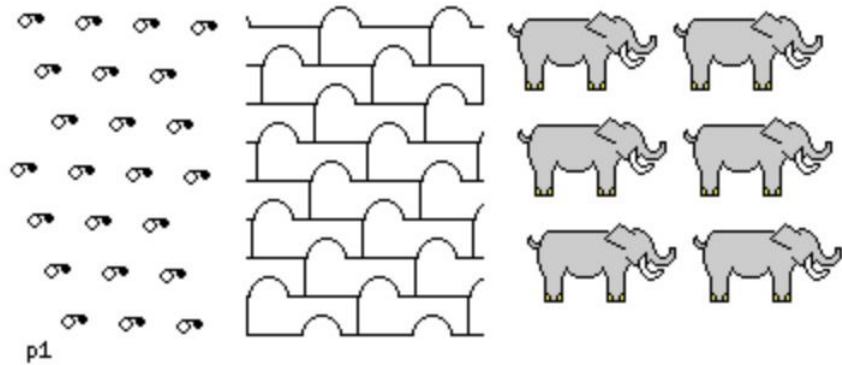
Isometries and Wallpaper Groups
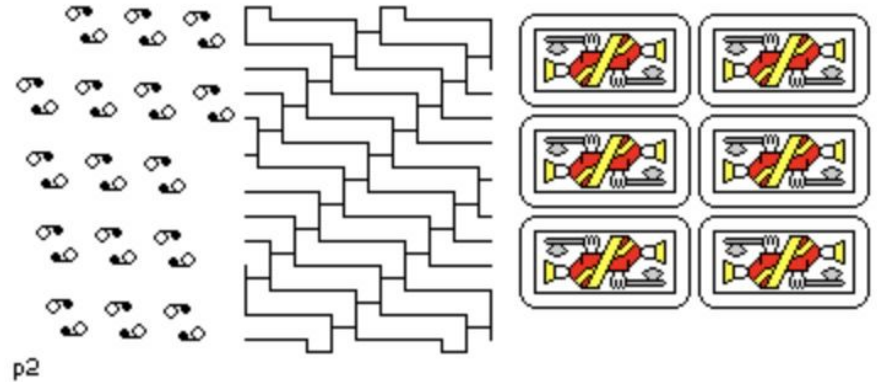

Left: patterns for the wallpaper groups

Isometries and Wallpaper Groups

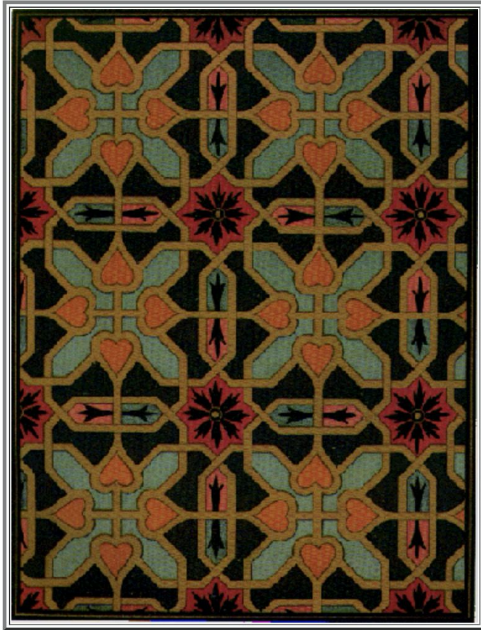[Illustrations of all 17 groups](#)

Group P1

Group P2

Wallpaper Groups in Islamic Art

This is a paper about wallpaper groups in the Alhambra.

From N. Simakoff: "Islamic Designs", p. 28
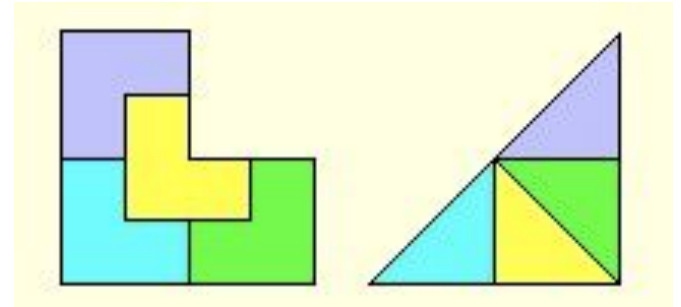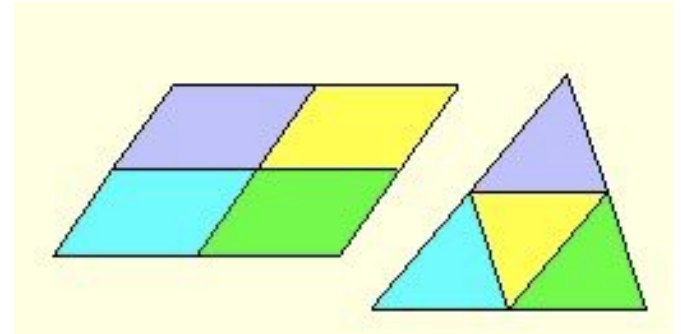Design from decorative painting on Bukhara architacture.

From N. Simakoff: "Islamic Designs", p. 40
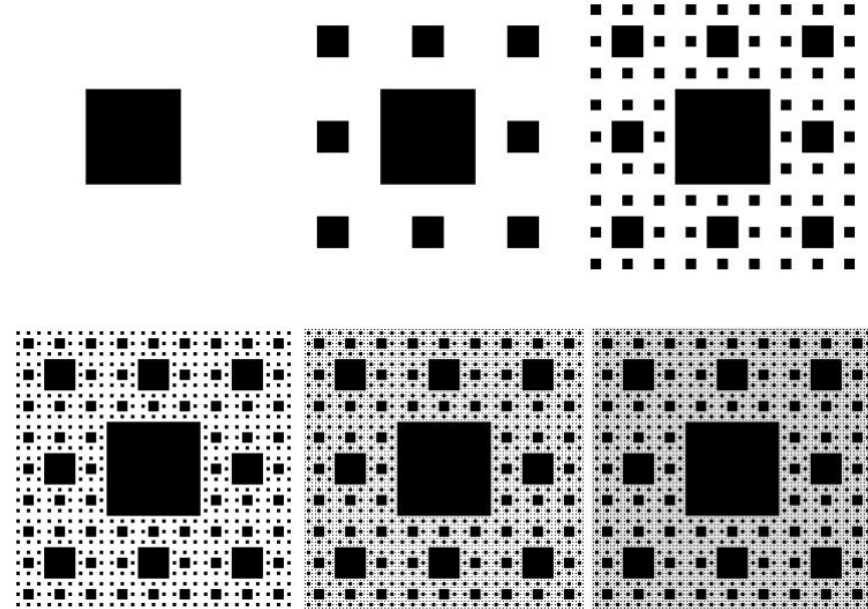Design from decorative painting on Bukhara architecture.

Tessellations and Fractals

- Tessellations as fractals: Reptiles
  - A reptile is called rep-n if it is dissected into n pieces.
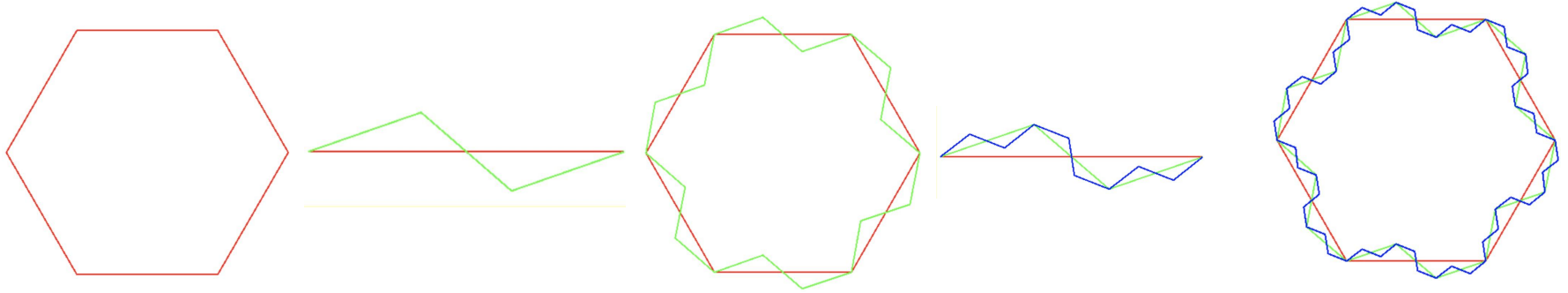
Tessellations and Fractals

- Tessellations as fractals: Reptiles
- Sierpiński Carpet
    - Upper left square: after 1st time
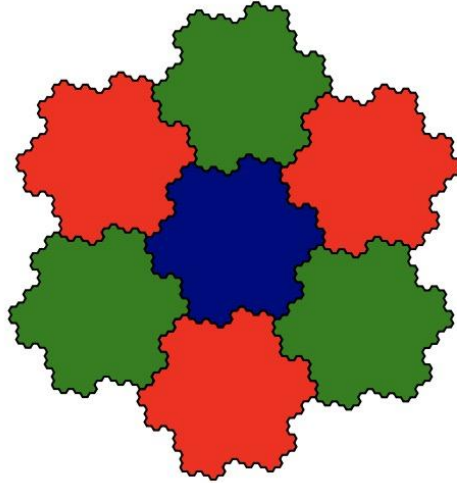   - Lower right square: after 6th time

Tessellations and Fractals

- Fractals as tessellations
  - Must have a non-empty interior
  - Example: **the Gosper Island**

Tessellations and Fractals

- After repeating the process on each side for seven times:

Seven copies of the Gosper island join together to form a larger shape.
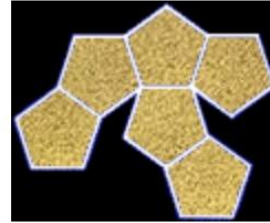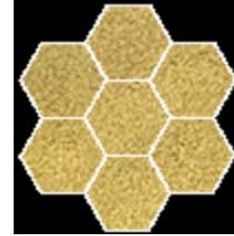
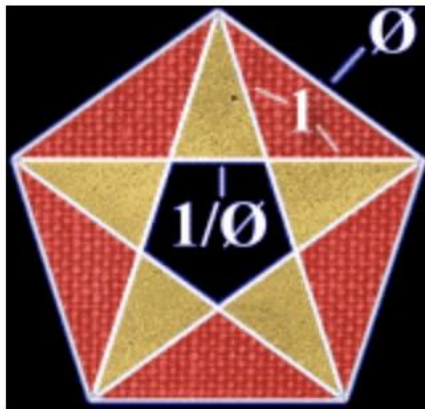# Penrose Tiles and the Golden Ratio
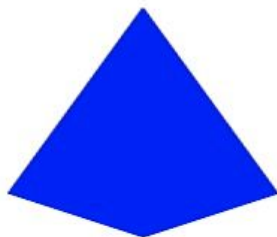

3 sides

4 sides

5 sides leaves gaps

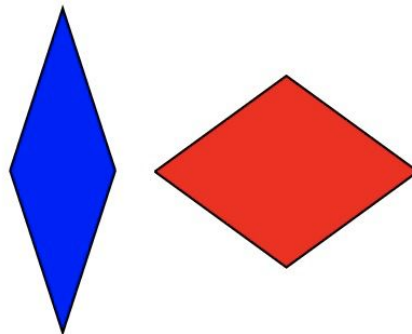6 sides

Penrose Tiles and the Golden Ratio

- Roger Penrose discovered that such a surface can be completely tiled in a symmetric manner with triangles and a pentagon
- Kite and dart tiling:
  - The ratio of the long side to the short side: the golden ratio involved
  - The ratio of the area of the kite to the dart
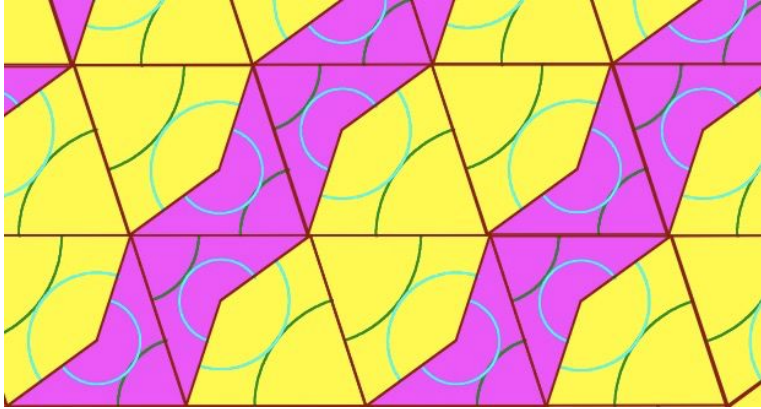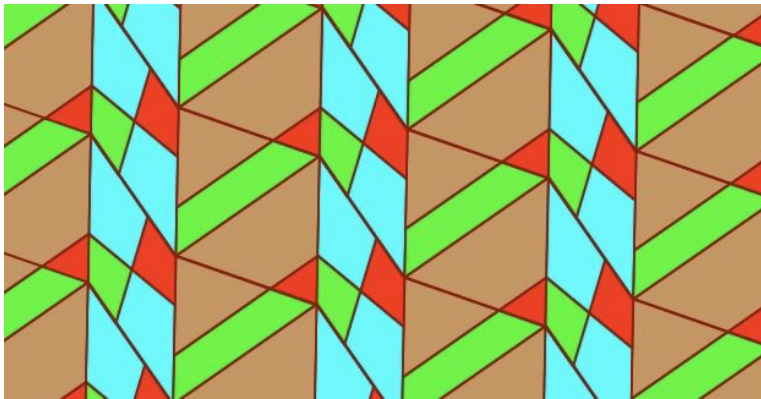

Kite

Dart

Penrose Tiles and the Golden Ratio

- The ratio of the quantities of the two tiles in resulting patterns approaches the golden ratio

References

http://mypage.concordia.ca/mathstat/pgora/m475/Gosper%20Island.html

https://www.goldennumber.net/penrose-tiling/

https://www.researchgate.net/profile/Yanxi_Liu/publication/2596375_Frieze_and_Wallpaper_Symmetry_Groups_Classification_under_Affine_and_Perspective_Distortion/links/0046351a3bc08c2384000000/Frieze-and-Wallpaper-Symmetry-Groups-Classification-under-Affine-and-Perspective-Distortion.pdf

http://www.quadibloc.com/math/tilint.htm

https://math.dartmouth.edu/~matc/math5.pattern/lesson5math.html