Magic Squares and Using Magic Series for Theory

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Abstract

In this paper, I discuss the basics of magic squares, building up to discuss recent developments in the enumeration of order 6 magic squares. I start off by going over the basic magic square terminology and some magic square history. From there, I move to discuss some foundational magic square theory: magic constants and magic series. I provide examples to demonstrate how magic series can be used to show either the existence or nonexistence of n-by-n magic squares in general. I finish by using the theory discussed to help explain Hidetoshi Mino's general methodology for enumerating all normal order 6 magic squares.

1 Introduction and Motivation

Magic squares are a small piece of recreational mathematics that I have been familiar with for a few years now. However, for a while, they stayed as just an amusing curiosity that I did not feel compelled to look into. It was not until I found (out of the blue) a magic square carved into the side of a religious building in Spain that my interest was piqued again. Once I was tasked with researching a math topic for this class, I quickly decided to delve into this piece of math that I had neglected for a while.

Although my research was primarily focused on magic squares and their history in art and religion, I stumbled upon quite a bit of theory that explained some of their properties and provided some explanations as to how magic squares really function. In my presentation, I did not go very deeply into the properties and theory behind magic squares, so I will discuss these more in this paper.

2 What are magic squares?

2.1 Brief Overview and Important Terms

A **magic square** is an n-by-n grid of numbers where the sums of the numbers in each row, each column, and both main diagonals are the same. We call this common sum among the rows, columns, and diagonals the **magic constant**. Theoretically, you could have all kinds of numbers inside the grid, but usually, the numbers are all positive integers. An n-by-n magic square is considered **normal** if it includes all the positive integers up to n^2. Much of the interesting theory of magic squares only occurs with these normal magic squares. Thus, when discussing magic squares, mathematicians often drop the normal classification and have readers assume that they are talking about normal magic squares unless otherwise stated. For clarity, though, I will keep appending the term 'normal' for this paper.

There are many different types of magic squares, but for now, I will introduce two important classifications of them: trivial magic squares and semimagic squares. A **trivial** magic square is one that has at least one repeated entry. Because one can easily construct a trivial magic square (think of a magic square consisting of just 1s), these kinds of magic squares are usually not very mathematically interesting and not heavily discussed. A **semimagic** square is one where all the row and column sums are equal, but one or both diagonal sums do not follow suit. Such a magic square can be created as a result of a failed attempt to make a normal magic square.

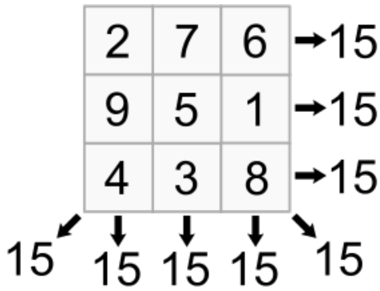
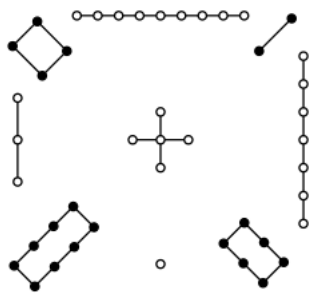


Fig. 1: An example of a (normal) magic square with magic constant 15.

An important property of any magic square is that the square remains magic whenever you (1) rotate it by 90 degrees one or more times, (2) reflect them horizontally or vertically, or (3) combine these two actions. In other words, magic squares remain magic when transformed by any of the 8 elements of D_4, the symmetry group of a square. This leads to a notion of equality among magic squares: if you can transform a magic square using an element of D_4 to get another magic square, then those two magic squares are in the same equivalence class. This equivalence relation becomes quite important when dealing with the enumeration of magic squares, e.g. the study of how many distinct normal magic squares there are (up to equivalence.) It is important to note, though, that while this equivalence relation works in general for magic squares of any order, different orders might have other isomorphisms between magic squares that come from a different set of transformations (which likely include at least some of the rotations and reflections mentioned above) [1].

2.2 Some Brief History

Magic squares have been around for quite a while. Although we do not know exactly when the first magic square was created, the “earliest clear reference” to any magic square shows up in the first-century Chinese book *Da Dai Liji* [2] . Specifically, the magic square that appeared in this book was the famous Luoshu magic square, an order 3 (3-by-3) square. According to legend, this magic square (or rather, the pattern of the numbers in the square) appeared on the shell of a magical turtle which emerged from the Luo river, which often flooded. The pattern of the numbers on the shell allowed the people to tame the Luo river and stop the flooding.



4	9	2
3	5	7
8	1	6

Fig. 2: The Luoshu square in its traditional Chinese representation (left) and a more familiar grid form (right).

From China, magic squares found their way into the Arab world via India, as evidenced by the long history of the magic square in India and potential links between Indian and Islamic magic squares [3] [4]. Magic squares seemed to have been “introduced to Europe through Spain” around the 11th century [3], and, like in other regions where magic squares were known, much of the study revolved around the construction and enumeration of magic squares. By the 16th century, magic squares were influential enough to be included in art. In Albrecht Dürer’s engraving *Melencolia I*, he includes a 4x4 magic square in the upper-right hand of the engraving, among other mathematical objects throughout the piece.

Magic squares remain a somewhat popular curiosity throughout the world today, though there are not many mathematicians who study them due to their perceived lack of usefulness. Nonetheless, there are a number of dedicated experts who continue and expand the tradition. Some have continued focusing on enumeration of magic squares, aided with the help of computer programs and algorithms. Others have developed new structures based on the idea of magic squares, such as magic cubes and magic hypercubes.

Fig. 3: *Melencolia I* by Albrecht Dürer and a close-up of the magic square.

For the sake of the paper, I will not focus on any of these new structures. Rather, I will introduce some foundational magic square theory, that of magic series, before going into the enumeration of order 6 normal magic squares, which has had an extremely recent development.

3 Foundational Theory

3.1 Magic Constants and Magic Series

This next section is based heavily off the work of Walter Trump [5], *many thanks to him.*

As mentioned previously, the **magic constant** is the common sum among the rows, columns, and diagonals in a normal magic square. The formula for the magic constant of an n-by-n square is:

$$M(n) = \frac{n(n^2+1)}{2}$$

This can be derived from the fact that (1) the sum of all integers 1 to m is $\frac{m(m+1)}{2}$ and (2) all integers 1 to n^2 exist in an order n normal magic square. Because there are n rows/columns in an order n square, you can divide the sum $1 + \cdots + n^2$ by n to get the common sum for the n rows/columns. After all, if you add all the row sums or column sums in a magic square, you have to get the sum $1 + \cdots + n^2$.

Within the magic square, we can look along the rows, columns, or diagonals to see the numbers we need to add together to get the magic constant. For example, in the Luoshu square, we can add $4 + 9 + 2$ in order to get the magic constant 15. Because 4+9+2 equals the magic constant, we call a sum like this a **magic series**. In the literature, these magic series are usually written in increasing order, i.e. $2 + 4 + 9$. Magic series are the building blocks of magic squares. To demonstrate this, let us think about what we need to construct an order 3 normal magic square.

In order to construct such a square, we need 8 distinct sums that all equal the magic constant 15: 3 for all the rows, 3 for all the columns, and 2 for both diagonals. In other words, we need 8 distinct magic series (where distinct means that not *all* the numbers in two series are the same.) Because we are trying to construct a normal square, the numbers within a sum must be distinct, positive integers. From here, we can start compiling all the possible sums of distinct positive integers that equal 15, e.g. all magic series or order 3:

1. $1 + 5 + 9$
2. $1 + 6 + 8$
3. $2 + 4 + 9$
4. $2 + 5 + 8$
5. $2 + 6 + 7$
6. $3 + 4 + 8$
7. $3 + 5 + 7$
8. $4 + 5 + 6$

As it turns out, there are *only* 8 possible magic series of order 3, which is exactly what we need to start constructing an order 3 normal magic square. All that needs to be done at this point is arranging these series within the square the right way so that all the series can work as either a row, column, or diagonal sum.

As I mentioned before, magic series are the building blocks of magic squares. Without a sufficient amount of these series, you cannot build a magic square. Indeed, we can use this idea to demonstrate that an order 2 normal magic square is impossible. A 2x2 normal magic square would only include the integers 1, 2, 3, and 4. The magic constant of such a square would also need to be $M(2) = 2(4+1)/2 = 5$. Further, there are 2 rows, 2 columns, and 2 diagonals in the square, so we would need 6 distinct magic series that equal 5. Let us compile all possible order 2 magic series:

1. $1 + 4$
2. $2 + 3$

There are only 2 of these magic series, nowhere near the 6 we need to build an order 2 normal magic square. Thus, a 2-by-2 normal magic square cannot be possible. To verify, one can check all $4! = 16$ arrangements of the integers 1-4 in a 2x2 grid and find that none of the arrangements equals a magic square.

3.2 Enumeration of Normal Magic Squares and Recent Developments

A natural question to ask after learning about magic squares and some of the theory behind them is: "How many normal magic squares of order n are there?" Usually, when magic square theorists ask this question, they include the condition 'up to equivalence,' i.e. not counting the squares you can get from rotations and reflections, as mentioned in the beginning of this paper. It may surprise you to learn that, although magic squares have likely existed for at least 2,000 years, mathematicians have only confirmed the exact number of normal magic squares up to order 5. Specifically, [6] documents the number of distinct order n normal magic squares for n 1 through 5, compiled in a table below:

Order of normal magic square	# of distinct squares (up to equivalence)
1	1
2	0
3	1
4	880
5	275,305,224

It should be noted that the order 5 enumeration value was calculated back in 1973 by Richard Schroeppel, so in terms of exact values for the number of normal magic squares of any order, this is the most recent confirmed value [6]. We do have estimates for the number of normal magic squares of order 6 and higher, though (see [7], [8]).

4 Hidetoshi Mino's Enumeration of Order 6 Magic Squares

4.1 A Very Recent Development

That being said, within the past year, there has been a breakthrough in the enumeration of 6x6 normal magic squares. Although the value is unconfirmed and needs to be thoroughly vetted due to the amount of computation involved, Hidetoshi Mino of the University of Yamanashi in Japan [9] has calculated that, as of February 17, 2024, there are **17,753,889,197,660,635,632** normal magic squares of order 6. Mino made 2 other estimates over this past year, but had found errors in the computation that were corrected. This result is backed up by a previous estimate [10] of $(1.7745 \pm 0.0016) \cdot 10^{19}$. Although Mino had to use an incredible amount of computational power to find this result ("hundreds of GPUs" and "about 80,000 hours [of computation time]" [9]), he was still able to reduce the amount of counting necessary by using a clever representation of magic squares as binary numbers.

4.2 Mino's Methodology for Enumerating 6x6 Normal Magic Squares

Once again, magic series are the key to solving this problem. We can utilize the binary representation of magic series to help us out. To be specific, a **binary representation** of a set of distinct positive integers $\{a_1, a_2, a_3, \cdots\}$ is a value equal to $2^{a_1-1} + 2^{a_2-1} + 2^{a_3-1} + \cdots$. For example, if we treat the magic series $2 + 4 + 9$ as a set (e.g. $\{2, 4, 9\}$), then we can represent the magic series as the binary integer $\texttt{1 0000 1010}_2$. Because the binary representations of the magic series we are dealing with are simply integers, we can define an order among magic series and compare them so that we can find the 'largest' magic series, that is, whichever magic series has the largest binary representation.

Mino also utilizes the concept of the complement of a magic series. If you take an order n magic series and replace every number x with n^2+1-x, you still have a magic series. The new series is the **complement** of the original magic series. For example, the order 4 magic series $\{5, 16, 2, 11\}$ has a complement $\{12, 1, 15, 6\}$. What is nice about the binary representation of a magic series is that you can simply reverse the bits in the representation, or rather, read the bit string backwards, to get the complement of the magic series. Specifically, we have that the complement of $\texttt{1000 0100 0001 0010}_2$ is $\texttt{0100 1000 0010 0001}_2$. This ultimately makes finding the complement of both a magic series and a magic square much easier.

With all this, Mino defines the **representative magic series** of a magic square as "the largest magic series which forms a row, a column, the complement of a row, or the complement of a column" [11]. If we take the following magic square:

16	12	1	5
7	3	14	10
9	13	4	8
2	6	15	11

then the representative magic series for this square would be $16 + 13 + 3 + 2$, which is the complement of the third column. As the word representative implies, Mino found that we can classify magic squares by their representative magic series [11]. This classification is not unique to each magic square; rather, Mino uses this classification to easily divide all the magic squares he needs to count into subsets corresponding to these representative magic series. This strategy helped him to divide the work among the hundreds of GPUs he used. Further, "this classification is invariant under rotations, reflections, M-transformations, and the complement transformation" [11]. By **M-transformations**, Mino is referring to the transformations that simultaneously permute rows and columns to preserve the numbers in each diagonal (the positions of the numbers within the diagonal may change [12]). These M-transformations altogether yield 24 different squares [13], so the counts that Mino got simply had to be multiplied by 24 to get his exact value.

All in all, below is a very simplified description of the procedure Mino took to count the squares for one subset corresponding to a representative magic series:

1. Generate all order 6 magic series (32,134 of them).
2. Since 14 magic series are required for an order 6 magic square (12 for rows and columns, 2 for diagonals), use the representative magic series and pull together 13 more magic series.
3. Ensure that these 14 magic series can be arranged to create a magic square not already produced. If so, add to the count for this representative magic series.
   - In this step, Mino optimizes for efficiency by not explicitly producing magic squares that are duplicates (obtained from rotations, reflections, M-transformations, and the complement transformation) of any square already produced.
4. Repeat steps 2-3 until all combinations of the representative magic series plus 13 other distinct series are exhausted.
5. Multiply the count by 24 to get the number of order 6 normal magic squares corresponding to the representative magic series we are working with (up to rotations and reflections.)

5 Reflection

After giving my original presentation on magic squares, I was actually surprised to learn that many of my classmates had not heard about magic squares until my presentation. During my presentation, I even prefaced my explanation of magic squares by saying: "You probably already know how these work, but…" Ultimately, I am glad that I was able to teach my peers about a new math concept, and that my peers seemed to like the connections I made to art and architecture. One big lesson I definitely learned from this process is that I need to pace myself better when it comes to doing research and actually coming up with the presentation. I was rushing too much right before the presentation was due. I also hope to interact more with the audience next time, such as having them answer a question relevant to the presentation.

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