

# Voronoi Diagram

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## Introduction

Voronoi Diagram is a space partitioning algorithm that has various mathematical connections in fields including computer graphics, geophysics, epidemiology and meteorology. In this paper, we will begin by discussing the history of the Voronoi Diagram, followed by the definition and construction of the algorithm. We will then examine the Voronoi Diagram's vast connection to nature and survey through the numerous applications it has to offer, along with its potential that has yet to be discovered.

## History and Development

Voronoi Diagram was first considered by René Descartes in 1644. Descartes utilized the Voronoi Diagram to partition the universe into “vortices” for his research. Though Descartes did not formally define the Voronoi Diagram, his work presented a very similar concept [1].

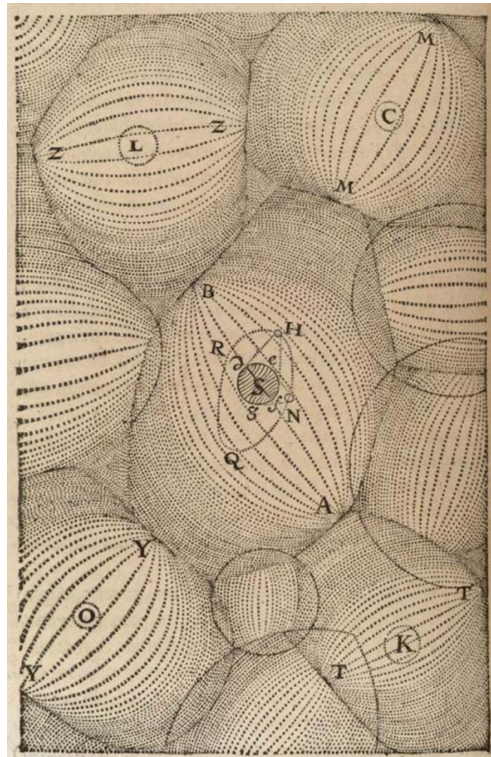


Figure 1. Descartes' work of universe “vortices”[1]

The formal definition of Voronoi Diagrams was first established from the work of two German mathematicians: Lejeune Dirichlet (1850) and Georgy Voronoy (1908) [2]. Dirichlet's work on positive quadratic forms contributed greatly to the development of Voronoi Diagram, making the algorithm also know as the Dirichlet tessellation. Dirichlet formalized the Voronoi Diagrams in the two and three dimensional space before 1859. By 1907, Voronoy formally defined the cases in higher dimensional spaces, giving the Voronoi Diagram its most commonly used name today[2].

## Definition and Basic Terminology

For the two-dimensional case, suppose a set of points  $S := \{p_1, p_2, p_3 \dots p_n\}$  in a plane. A Voronoi diagram partitions the plane into  $n$  **Voronoi regions** (also know as the Voronoi cells) with following properties:

- Each Voronoi region contains exactly one point  $p_i$ , also known as the **Voronoi site**. The boarder of two Voronoi regions is called **Voronoi edge**.
- For an arbitrary point  $q \notin S$  in the same Voronoi region as  $p_i$ , the distance from  $q$  to  $p_i$  will be no greater than the distance from  $q$  to  $p_j$ , where  $p_j$  is an arbitrary point in  $S$  [4].

For a more general formal definition that goes beyond a plane, consider a metric space  $X$  with distance function  $d$ . Let  $S$  be a set of indices and let  $(P_k)_{k \in S}$  be a ordered collection of nonempty subsets (Voronoi sites) in the space  $X$ . The Voronoi region  $R_k$  associated with site  $P_k$  is the set of all points in  $X$  whose distance to  $P_k$  is no greater than their distance to any other site, similar to the second property introduced previously. If the distance formula denoting the distance between point  $x$  and subset  $A$  can be written as  $d(x, A) = \inf\{d(x, a) \mid a \in A\}$ , then

$$R_k = \{x \in X \mid d(x, P_k) \leq d(x, P_j), j \neq k\} \quad [3]$$

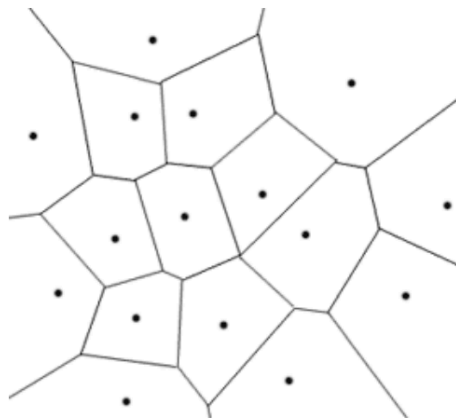


Figure 2. Voronoi Diagram in a 2D plane [4]

## Construction of Voronoi Diagrams

While we discuss the methods to construct a Voronoi diagram, we are mainly focusing on the two-dimensional case. The reason being that multi-dimension constructions of the Voronoi diagrams are predominantly done with the help of computer modeling.

Before diving into the procedures, notice that perpendicular bisectors will play an important role in constructing Voronoi diagrams.

- First, we will select three arbitrary sites  $P_1, P_2, P_3$  to form a triangle. Since the perpendicular bisectors of a triangle must intersect at a point, we take the perpendicular bisectors of each side of the triangle and extend them to an intersection.
- Remove the superfluous part of the bisectors after the intersection, we will result in a base Voronoi diagram of three points. Next, we will select another arbitrary site  $P_k$  and take its perpendicular bisectors to each of the previously selected sites.
- Everytime the newly drawn bisector hits an old Voronoi edge, we will remove its segment in any other Voronoi regions and set the remaining as the new Voronoi edge between  $P_k$  and the currently selected site from the previous diagram. Any old edges between the two sites shall be removed making the newly formed Voronoi edge the only segment between these two sites.
- Step two and three will be repeated until we traverse through all  $n$  sites on our given plane, giving us the resulting Voronoi diagram.

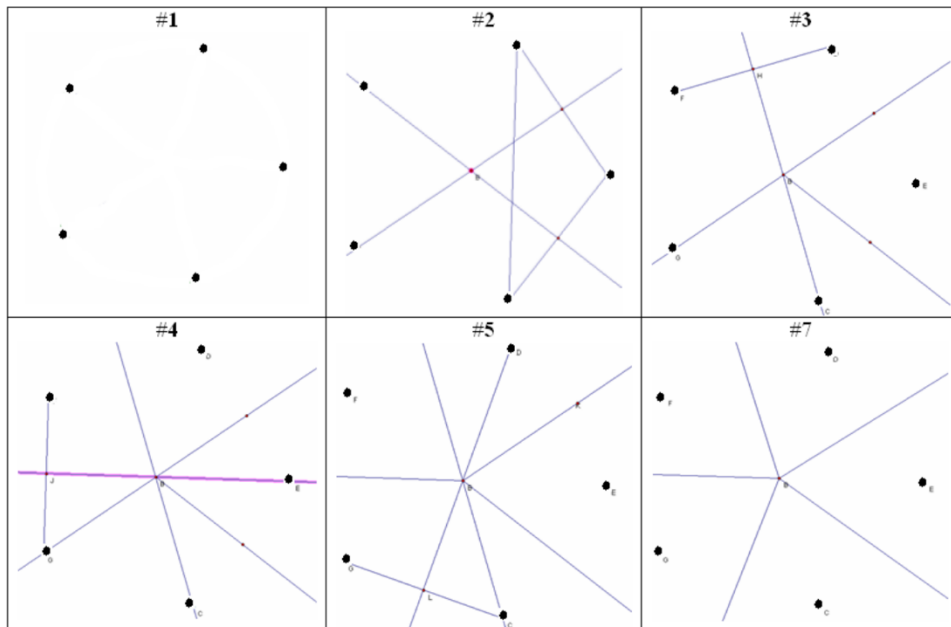


Figure 3. Construction of a Voronoi Diagram [1]

## Natural Connections of Voronoi Diagrams

Consider the venation of a leaf. Why does it grow up to have a Voronoi pattern? The answer is relatively simple if we break it down and compare leaves' growing process to the Voronoi algorithm.

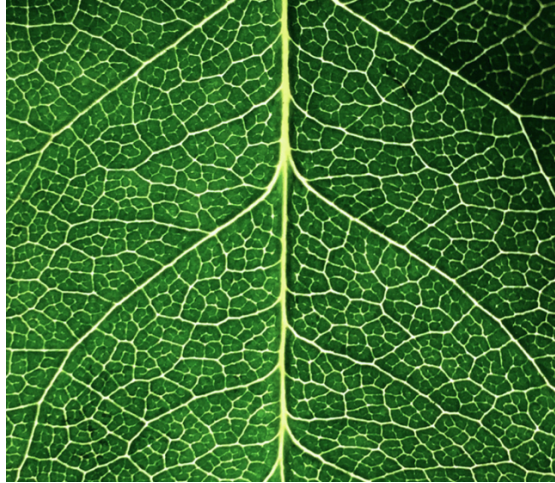


Figure 4. Leaf Venation [5]

The pattern of a leaf depends on the rate of growth of its tissues around its veins. The tissues that are next to the veins will have a much greater growth rate than tissues away from the veins. If we consider the pattern of a leaf as a Voronoi diagram, the veins will be the Voronoi edges. In each Voronoi region, the Voronoi site will be the point where the rate of growth is the slowest. While the leaf grows, the tissues on both side of the veins grow at an equal rate, making the Voronoi site (slowest growth rate point) be placed symmetric about every vein. As a result, the leaves will have a Voronoi pattern.



Figure 5. Insect Wings Pattern [6]

The Voronoi pattern does not only appear on leaves, it also has many other occurrences in nature. Insects' veins on their wings contain hemolymph and trachea to support their growth and emergence. It will also have a Voronoi pattern similar to the process of leaves' veination. The cracks on a dry ground also form a Voronoi pattern. The reason being that the water evaporate faster near the cracks than other surfaces. The fact that the Voronoi pattern is a result of the difference in rate of growth makes it one of the most common patterns in nature.

## Applications of Voronoi Diagrams

Let us consider the following problem. Suppose we want to estimate the customer flow of a group of shops in a given city. For all of these shops, assume that all quality factors including prices, products, services are equal [3]. We are able to infer that the customers will choose their preferred shop based on distance. In this case, each Voronoi region of the diagram constructed can be used as an estimator of the customer flow of the store represented by the Voronoi site within that region.

The Voronoi Diagram can be used to solve such problem and many others including hospital allocation, school area delimitation etc. In Melbourne, government school students will attend the nearest school measured by straight-line distance. The school zone delimitation will therefore result in a Voronoi diagram. One application of the Voronoi diagram can be related to the current Covid-19 pandemic situation. The algorithm can be used to correlate sources of infections. As one of the early examples of this application, John Snow(1858) implemented the Voronoi diagram to study the 1854 Broad Street cholera outbreak in Soho, England. His research demonstrated the strong correlation between residential areas on the map of Central London whose residents had been using a specific water pump, and the areas with most deaths due to the outbreak[3].

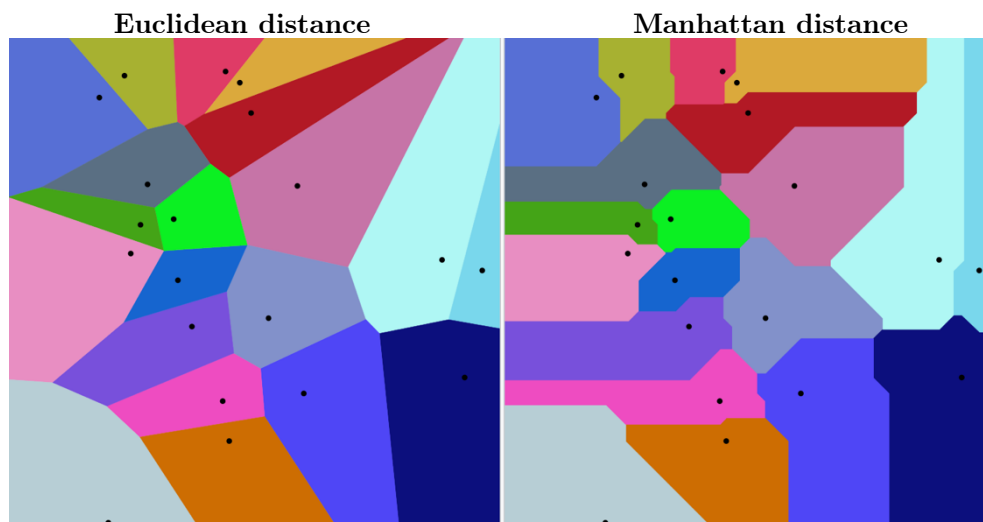


Figure 6. Voronoi Diagram solution to the customer flow problem[3]

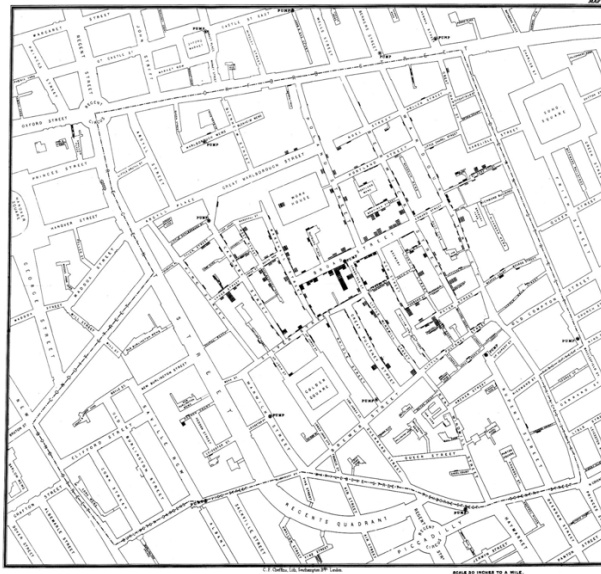


Figure 7. John Snow's diagram[3]

The Voronoi diagrams also have many application across disciplines. These applications are not limited to the algorithm itself. Some of the examples will take a particular part of the Voronoi diagram (ex. The Voronoi pattern), while others extend beyond the two-dimensional support (3-D cell modeling).

- In the natural science field, Voronoi diagrams are used to model biological structures including cells and atom architectures.
- Voronoi diagrams are utilized in ecology to study the growth of forests and to predict forest wildfires.
- Voronoi diagrams are used to generate adaptative smoothing zones on images and to add fluxes on each one. It helps to produce constant signal-to-noise ratio on all images, mainly used in the field of astrophysics[3].
- Voronoi diagrams are used in computational physics to calculate profiles of an object with Shadowgraph and proton radiography in High energy density physics[3].
- In mining, the Voronoi diagrams can be used to estimate the mining value and to plan drillholes as they act as the site point in the diagram.
- In networking, Voronoi diagrams can be used to derive the capacity of wireless networks[3].
- Voronoi diagrams can be used to study insect or animal territorial behaviors. Namely the Bark Beetle was known to have a feed pattern similar to Voronoi regions so that they prevent intra-species competition for the same resource[1].

The applications listed above are only a glimpse of what Voronoi diagram's capability. The algorithm has potential to be applied in science, engineering, art, or even humanities. With the help of computer modeling, we are able to take the Voronoi diagram into multi-dimension and further expand its usage in the near future.

## **Conclusion**

Although we have already discovered so many areas that the Voronoi diagrams can be used for, there are still potential of this algorithm that is yet to be explored. The use of Voronoi diagram in the information science field or even cryptology; the investigation of efficiency of different construction methods and reverse engineering processes; the reason why the Voronoi pattern is considered beautiful and even implemented in countless architectures. Overall, the Voronoi diagram is a simple yet complicated concept that excels in its versatility of usage, with enormous value embedded to be unfolded.

## REFERENCES

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