# Exploring the Relationship Between the Manipulation of the Stock Market and the Application of Mixed Strategy

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**Abstract**

Stock manipulation, or the artificial inflation or deflation of the price of a stock, is prevalent among the companies with small market capitalization. Basically, the manipulator, typically with abundant financial resources, has two options: to pump or to dump. An individual buyer also has two options: to buy or to short a shock. Therefore, we can inspect the situation on a microscopic level as a two-player zero-sum matching penny game. For convenience, we assume that both players are fully rational and make their decision simultaneously. We will prove that if the manipulator precisely pick his strategy based on a certain probability distribution interval, the long-term expected payoff of the individual trader would be negative, which means the manipulator would always earn money from an individual trader no matter which mixed strategy the trader chooses.

## 1 Introduction

### 1.1 The Main trading options for individual buyers

Generally, traders have two main strategies to choose. If the trader predicts that the price of a stock will rise in the future, the trader could choose to buy and earn money if the price rises. The player could also choose to short-sell a stock and benefit from the decline of the stock price. Specifically, since the price of a stock can not decline below zero and it can rise to theoretically infinity, if the trader buys a stock and the price declines, the worst case would be that the trader loses the original principal. However, if the trader short-sells a stock and the price ascends, the trader could have unlimited loss [1]. We will consider these conditions when we design the model.

## 1.2 The Mechanics of Manipulation

In stock market trading, a manipulation happens when an individual or a financial institution manages abundant resources in terms of both the capital and information. In short, by controlling several accounts in the name of different account owners, the manipulator is able to create an artificial trading environment where the demand and supply quantities of the stock are under the manipulator's control[2].

Although the action of buying a stock itself is as same as buying daily necessities, there are numerous ways to evaluate the value of securities like stocks. Generally, most people believe that the price of the stock of a company can reflect the companies' value and potential growth [3]. People who are unable to analyze financial reports of companies make decision based on financial news [4]. Therefore, due to this assumption, the manipulator would promulgate the news that boasts a company's earning potentials when the manipulator choose to rise the price so that people can buy the overvalued stocks from them thereafter. Conversely, if the manipulator decides to make money from buying some undervalued stocks, he or she would decrease the price and convince other people to short the company's stocks [1].

## 1.3 Problem of the discrepancy between expected payoff of different participants

We suppose that there are only one manipulator and one trader. How does manipulator benefit from the manipulation of the price of stocks? If both the individual trader and the manipulator each have two options, it would lead to four different outcomes. Two of them are beneficial for individual trader and detrimental to the manipulator. Another two outcome would result vice versa. Is it possible for the expected payoff of the manipulator to be always higher than that of the individual trader?

## 1.4 Method and Result

For convenience, we will evaluate the problem above on a microscopic level. We assume that an individual trader, (player 1), and a stock manipulator, (player 2), play a zero-sum matching penny game. Each player has two strategies: to show the head or to show the tail. They make decisions simultaneously. We will show the payoff matrix of the four possible outcomes, which we will explain in detail in the Methods section. Among the four possible outcomes, each player would benefit from two outcomes and be frustrated by the other two outcomes. Zero-sum means that the gain of one player is another player's loss.[5] For example, in our example below, if both players choose to show head, the trader's payoff is $+3$ and the manipulator's payoff is $-3$. In reality, this matches with the situation that the trader buys a stock and manipulator raises the price of the stock.

Using properties of expected value and decreasing function, we showed that there will be an interval of probability distribution for the manipulator to make the expected payoff of the trader always less than zero as long as the manipulator design the game satisfying several conditions, which we will discuss in the Method section. For our specific example, the interval is between 1/3 to 2/5. In other words, the expected payoff of the trader is always less than zero if the manipulator chooses to show head with a probability between 1/3 and 2/5, (thus, the probability of choosing tail would be correspondingly between 3/5 and 2/3). This is the concept of mixed strategy.

## 1.5 Some theorems and terms

### 1.5.1 Pure Strategy

Let $S$ be a set of strategies that a player can choose from. A pure strategy determines that a player would play a strategy $s \in S$ for any strategies that other players would choose [5].

### 1.5.2 Mixed Strategy

Instead of choosing a strategy with probability of 1, a player chooses a strategy according to a probability distribution [5].

### 1.5.3 Expected Value

The expected value is determined by multiplying each of the possible outcomes by the likelihood that each outcome will occur and summing all of the values [6].

### 1.5.4 Decreasing function and mean value theorem

For any $f(x)$ to be a decreasing function, we need to use the mean value theorem [7] and check three conditions: For an interval $[a, b]$, where $b > a$,

1. $f$ is continuous on $[a, b]$.
2. $f$ is differentiable on $(a, b)$.
3. There is a point $c \in (a, b)$ such that $f\prime(c) = \frac{f(b)-f(a)}{b-a}$

## 2 Materials and Methods

### 2.1 Experimental Design

We assume that the game is a zero-sum game so that a player's gain is another player's loss. The gain of one player is another player's loss. Each player has two strategies to decide, which leads to four possible outcomes:

- Let $m$ and $n$ be positive rational numbers, where $m \neq n$ and $m < n$.
- *Outcome* 1: Both players show head. Then, the trader will earn $(m + n)$ dollars from the stock price manipulator.
- *Outcome* 2: The trader shows the head and the manipulator shows the tail. Then, the trader will lose $n$ dollars to the stock price manipulator.
- *Outcome* 3: The trader shows the tail and the manipulator shows the head. Then, the trader will lose $n$ dollars to the stock price manipulator.
- *Outcome* 4: Both players show the tail. Then, the trader will earn $(n - m)$ dollars from the stock price manipulator.

We can condense the information into a 2x2 payoff matrix showing in figure (1):

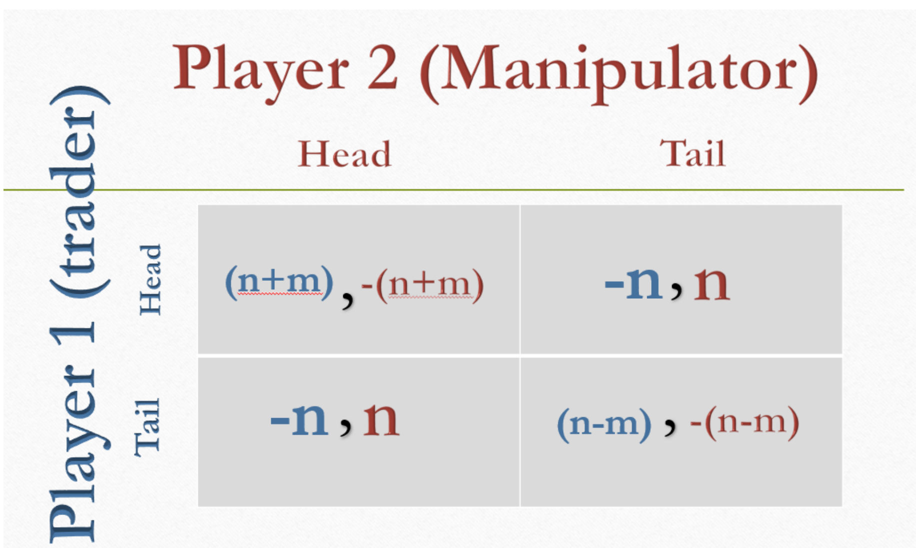


Figure 1: The Payoff Matrix of the general case

Since the trader's benefit from buying stocks is theoretically unlimited and and the potential loss ceases when the price drops to zero, we design the expected payoff to be ($n$+$m$,$-$($n$+$m$)) for the head-head situation. Correspondingly, because the price can not decline below zero,the trader's expected benefit from short-selling a stock is limited and the potential loss is unlimited. The expected payoff for the tail-tail situation is ($n-m$, $-(n-m)$). To make the game seemingly fair, the sum of both players' payoff under the two circumstance that they could earn money is equal to $2n$. We will prove that as long as the manipulator's payoff is evenly distributed as n for the two outcomes that in favor of him or her, the long-term expected payoff of the trader is always less than zero if the manipulator chooses his strategy within a range of probability. We will prove the general case and use the special case as an example to illustrate the application.

## 2.2 Analysis

Let $u_1$ be the expected payoff of the trader. Let $\sigma_1$ be the probability that player 1 chooses to show the head. Then, the probability of the trader showing the tail is (1 - $\sigma_1$). Let $\sigma_2$ be the probability that player 2 chooses to show the head. Similarly, the probability of the manipulator showing the tail is (1 - $\sigma_2$). Thus, $0 \leq \sigma_1, \sigma_2 \leq 1$.

Therefore, the expected payoff of the trader would be the sum of the trader's payoffs under the four circumstances times the probability that both players would choose the given situation [6] :

$$u_1 = (n+m)\sigma_1\sigma_2 + (-n)\sigma_1(1-\sigma_2) + (-n)(1-\sigma_1)\sigma_2 + (n-m)(1-\sigma_1)(1-\sigma_2) \tag{1}$$

After simplifying, we have:

$$u_1 = (4n)\sigma_1\sigma_2 + (m-2n)\sigma_1 + (m-2n)\sigma_2 + (n-m) \tag{2}$$

Set $u_1$ to be less than 0, and solve for $\sigma_2$:

$$u_1 = [(4n)\sigma_1\sigma_2 + (m-2n)\sigma_1 + (m-2n)\sigma_2 + (n-m)] < 0 \tag{3}$$

$$[(4n)\sigma_1\sigma_2 + (m-2n)\sigma_2] < [(2n-m)\sigma_1 + (m-n)] \tag{4}$$

$$\sigma_2[4n\sigma_1 + m - 2n] < [(2n-m)\sigma_1 + (m-n)] \tag{5}$$

Since $m < n$ and $\sigma_1$ is unknown, we could not tell whether $[4n\sigma_1 + m - 2n]$ is positive or negative. Therefore, we will divide it into two cases in order to solve for $\sigma_2$.

### 2.2.1 Decreasing function proof

Before we continue to analyze the two cases, we need to prove that $f(\sigma_1) = \frac{[(2n-m)\sigma_1+(m-n)]}{[4n\sigma_1+m-2n]}$, where $0 \leq \sigma \leq 1$, is a decreasing function using mean value theorem [7]:

Since

$$\lim_{\sigma_1 \to +\frac{2n-m}{4n}} f(\sigma_1) = \infty \quad and \qquad \lim_{\sigma_1 \to -\frac{2n-m}{4n}} f(\sigma_1) = -\infty \tag{6}$$

There is a vertical asymptote at $\sigma_1 = \frac{2n-m}{4n}$. Notice that $m < n$, so $0 < \frac{2n-m}{4n} < 1$. Therefore, we split the domain of $f$ into two intervals: $[0, \frac{2n-m}{4n})$ and $(\frac{2n-m}{4n}, 1]$.

1. $f(\sigma_1)$ is continuous on both intervals.

2. We can calculate the derivative of $f(\sigma_1)$ to see that

$$f\prime(\sigma_1) = -\frac{m^2}{(4n\sigma - 2n + m)^2} \tag{7}$$

$f(\sigma_1)$ is differentiable on both intervals $[0, \frac{2n-m}{4n})$ and $(\frac{2n-m}{4n}, 1]$.

3. For any $c \in [0, \frac{2n-m}{4n})$ and $(\frac{2n-m}{4n}, 1]$, $f\prime(c)$ is negative as you can see from equation(9).

Hence, we conclude that the $f(\sigma_1)$ is a decreasing function on both intervals.

### 2.2.2 Two Cases

Let us continue to solve for $\sigma_2$

Case 1: $[4n\sigma_1 + m - 2n]$ is positive, $\sigma_1 > \frac{2n-m}{4n}$.

$$\sigma_2 < \frac{[(2n-m)\sigma_1 + (m-n)]}{[4n\sigma_1 + m - 2n]} \tag{8}$$

As we have proven in the previous subsection, the function on the right side of the $<$ is a decreasing function. In order to obtain the minimum value of $\sigma_2$, we pick $\sigma_1 = 1$ because $f(\sigma_1)$ is a decreasing function and $\frac{2n-m}{4n} < \sigma_1 \leq 1$ in this case. Thus, we have:

$$\sigma_2 < \frac{n}{2n+m} \tag{9}$$

Therefore, $\frac{n}{2n+m}$ is the upper limit of our intended interval for $\sigma_2$.

Case 2: $[4n\sigma_1 + m - 2n]$ is negative, $\sigma_1 < \frac{2n-m}{4n}$.

$$\sigma_2 > \frac{[(2n-m)\sigma_1 + (m-n)]}{[4n\sigma_1 + m - 2n]} \tag{10}$$

Similarly, $f(\sigma_1)$ is a decreasing function and $0 \leq \sigma_1 < \frac{2n-m}{4n}$ in this case. Thus, we pick $\sigma_1 = 0$ to obtain the maximum value of $\sigma_2$. Thus, we have:

$$\sigma_2 > \frac{n-m}{2n-m} \tag{11}$$

Therefore, $\frac{n-m}{2n-m}$ is the lower limit of our intended interval for $\sigma_2$.

To test the upper and lower bounds, we have:

$$\frac{n}{2n+m} - \frac{n-m}{2n-m} = \frac{m^2}{(2n+m)(2n-m)} \tag{12}$$

Since $n$ and $m$ are both positive and $n > m$, $\frac{m^2}{(2n+m)(2n-m)}$ is positive. Therefore, $\frac{n}{2n+m} > \frac{n-m}{2n-m}$.

Hence, if $\sigma_2 \in (\frac{n-m}{2n-m}, \frac{n}{2n+m})$, $u_1$, the expected payoff the player 1 is always less than 0 no matter what $\sigma_1$ player 1 chooses.

### 2.3 Example:

Here is an example in order to elucidate the concept.

Let $n = 2$, $m = 1$.

We will obtain a payoff matrix as figure(2) shows:

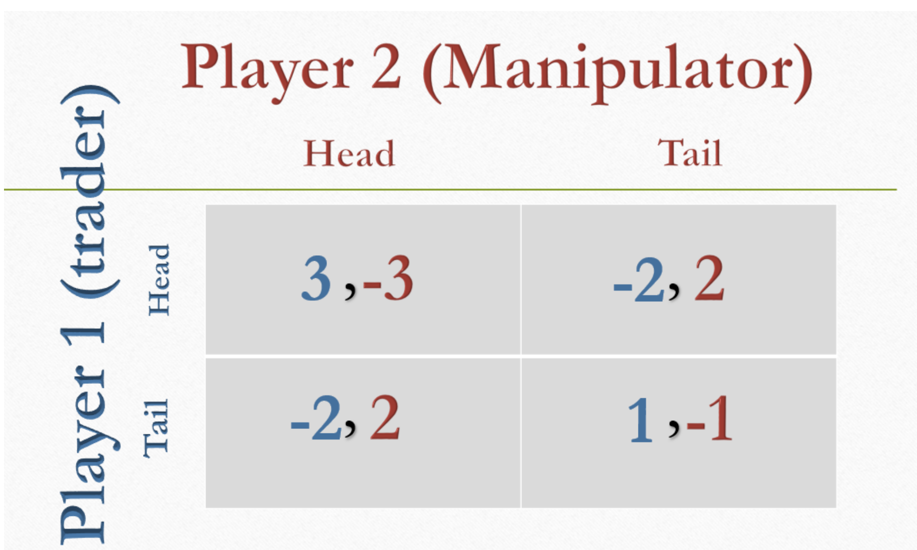


Figure 2: The Payoff Matrix of example

Let $u_1$ be the expected payoff of the trader. Let $\sigma_1$ be the probability that player 1 chooses to show the head. Then, the probability of the trader showing the tail is (1 - $\sigma_1$). Let $\sigma_2$ be the probability that player 2 chooses to show the head. Similarly, the probability of the manipulator showing the tail is (1 - $\sigma_2$). Thus, $0 \leq \sigma_1, \sigma_2 \leq 1$.

According to equation (1), the expected payoff of the trader is:

$$u_1 = 3\sigma_1\sigma_2 - 2\sigma_1(1-\sigma_2) < 0 \tag{13}$$

We set $u_1$ to be less than 0 and solve for $\sigma_2$.

According to equation (6), there is a vertical asymptote at $\sigma_1 = \frac{3}{8}$. $f(\sigma_1) = \frac{3\sigma_1 - 1}{8\sigma_1 - 3}$ is a decreasing function on both $[0, \frac{3}{8})$ and $(\frac{3}{8}, 1]$, which we have proven in section 2.2.1.

According to equation (9), $\sigma_2 < \frac{2}{5}$

According to equation (11), $\sigma_2 > \frac{1}{3}$

Hence, we can conclude that if the manipulator chooses to show the head (raise the price) within a probability interval $[1/3, 2/5]$, the individual trader's long-term expected value of payoff is negative for whatever $\sigma_1$, or the probability of showing head (to buy the stock) the trader chooses.

# 3 Results

We designed the model to be a 2v2 zero-sum matching penny game between an individual trader and a stock price manipulator. Both players have two strategies, to show head or to show the tail. Players will show their coins simultaneously. The game is seemingly fair since the sum of expected payoff for the two outcomes that is favorable to the trader is $(n+m)+(n-m) = 2n$, and that of the manipulator is $n+n = 2n$, where $n$ and $m$ are positive rational numbers and $n > m$. To investigate the integrity of the game, we used the mean value theorem, decreasing function property, and the expected value in mixed strategy games, and concluded the following results:

- If the probability that the manipulator would choose to show the head is between the interval $(\frac{n-m}{2n-m}, \frac{n}{2n+m})$, the long-term expected payoff of the individual trader is always less than 0 no matter how the trader distributes his mixed strategy.

- Conversely, since $f(\sigma_1)$ is a decreasing function for both $0 \leq \sigma_1 < \frac{2n-m}{4n}$ and $\frac{2n-m}{4n} < \sigma_1 \leq 1$, we obtained the upper bound of the interval for $\sigma_2$ using $\sigma_1 = 1$ and the lower limit of the interval using $\sigma_1 = 0$. Hence, we have covered all the mixed strategy of the individual trader (player 1) and there is no way for player 1 to adjust his or her mixed strategy to retaliate player 2.

# 4 Discussion

## 4.1 About the mixed strategy Nash Equilibrium

I did not introduce the concept of mixed strategy Nash Equilibrium because it is not quite relevant to our main points. Someone was asking about the mixed strategy Nash Equilibrium of this game during our discussion in class. We can have a detailed discussion here:

Let $\sigma_2$ be the probability that the manipulator would choose to show the head and $1-\sigma_2$ that the manipulator would show the tail.

Let $u_h$ be the expected payoff of showing the head for player 1:

$$u_h = (n+m)\sigma_2 - n(1-\sigma_2) \tag{14}$$

and $u_t$ be the expected value of showing the tail for player 1:

$$u_t = (n-m)(1-\sigma_2) - n\sigma_2 \tag{15}$$

Once the game reaches Nash Equilibrium, nobody can obtain a higher payoff by deviating from their probability distribution at the time [5]. Therefore, the trader's expected payoff of choosing the head would equal to that of choosing the tail for $\sigma_2$ at Nash Equilibrium. We could set $u_h = u_t$:

$$(n-m)(1-\sigma_2) - n\sigma_2 = (n+m)\sigma_2 - n(1-\sigma_2) \tag{16}$$

$$\sigma_2 = \frac{2n-m}{4n} \tag{17}$$

Similarly, we can prove that $\sigma_1 = \frac{2n-m}{4n}$. Notice that this is also the vertical asymptote that we calculated in equation (6). Plugging $\sigma_1$ and $\sigma_2$ into equation (1), we have the expected payoff of the trader as:

$$u_1 = \frac{-m^2}{4n} \tag{18}$$

Notice that $\frac{-m^2}{4n}$ is negative since $m$ and $n$ are both positive. Our interval did not include the vertical asymptote at $\sigma_1 = \frac{2n-m}{4n}$, but this calculation proves that the trader's expected payoff is still less than 0 even at the mixed strategy Nash Equilibrium.

#### 4.1.1 Nash Equilibrium when m = n

Let $m = 0$ and $n > 0$ According to equation (17), $\sigma_2$ is $\frac{1}{2}$. According to equation (18), the expected payoff of the trader, $u_1$ is 0. Thus, both players will evenly distribute their probability over the two strategies and the integrity of the game is secured.

### 4.2 My own story with Game Theory

I learned the traditional Chinese board game, Go Game, starting from age five. I barely had a chance to play Go Game with someone in my real life after I came to the United States. After I have no longer participating competitions, in recent years, it reminds of the enjoyable summers in my childhood.

For the first time, it was when Alpha-Go defeated the world champion Lee Sedol. All the Go Game players would not believe it because Lee Sedol was reputed as one of the most brilliant and meticulous artist. His discernment to envision the big picture and his ability of calculating the expected payoff the each decision were insurmountable. On the other hand, Go Game software before Alpha-Go was so dumb that an intermediate player can outwit it.

Several years later, Game Theory helped me solve a mystery that have been perplexed me over a decade. My habit of decision making when I played Go was to consider as more possible outcomes as I could. I could sit there all day and try to formulate a best response for both players. I used to encountered a certain dilemma, which I called "infinite Matryoshka doll". "If I choose strategy X, my opponent would choose Y. If I have known that my opponent would choose Y, I would have chosen Z. If my opponent would have known that I would have known that he would have known..." It just seemed like an endless Matryoshka doll. You will always find a smaller new doll after you open it. Since I was searching for the best response, I have to assume that my opponent and I are both fully rational. Now game theory tells me that I have been searching for the pure strategy Nash Equilibrium, which did not exist for the dilemma. However, there will be a pure strategy Nash Equilibrium if we choose strategies based on a certain probability distribution [5].

In short, game theory is art to me. It enlightens me to visualize the connection between the instincts that we have been trained in the past and the theoretical knowledge that we can utilize now.

### 4.3 Peer Response about the Presentation

My work received plenty of compliments and suggestion from the audience. I appreciate everyone's comment and advice. Most audience expressed their fascination with the topic and they also claimed that the decreasing function part might not be clear enough. In my next topic, I would further simplify the calculation part. I believe that presentations should consider the balance of professionalism and captivation. More importantly, one should prioritize the enthrallment with the techniques like story-telling because we can include the throughout and rigorous proofs in the paper.

### General

My thanks to Professor Chi-Kwong Li for suggesting the direction of generalizing the idea of assigning values for the payoff matrix.

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