Analysis on Three Common Western Tuning Systems

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Throughout the history of Western Music, music theorists have been trying to develop alternative ways to tune musical instruments to achieve a more pleasant playing and listening experience. These experiments resulted in different tuning systems, i.e. temperaments, which determines the frequency of each note.

Among various tuning systems, three temperaments were commonly used in Western Music at some point of time: Pythagorean Tuning, Quarter Comma Meantone Temperament, and 12 Tone Equal Temperament[1], in chronological order of each temperament's dominant time period.

In this paper, we will use algebra to examine these three tuning systems. We will consider their construction and interval ratios, and discuss some strengths and weaknesses of each tuning system.

Musical Background

Just Intervals. Music, especially melodic music, is carried by soundwave. Therefore, like any repeated signal with a consistent frequency, musical pitches contain harmonic series, waves with frequencies that are integer multiples of the base frequency. Theoretically, there are infinite harmonics of a given base frequency, but in practice, harmonics generally get weaker as they get higher, and become unobservable after a certain point. The ratios between the harmonics are called "just intervals". Because harmonics are always integer multiples of the base frequency, just intervals always from rational ratios, and because the harmonic series is theoretically infinite, there are theoretically infinitely many just ratios. Ancient Greeks discovered that two pitches whose base frequencies form simple integer ratios sound pleasant when played together, which is consistent with the popular mathematic system that only includes rational numbers. Today we understand that these pitch combinations sound pleasant because they form simple just intervals, and their harmonic series naturally overlaps with each other [2]. Some important just intervals for this paper are Perfect Fifth (3:2), Major Third (5:4), and Octave (2:1)

Octave. Due to the simplicity of the 2:1 ratio, pitches that are octaves apart have highly overlapping harmonic series and are often considered to be the same note. Therefore, a pitch with a base frequency of 220 Hz is consider the same note as pitches with the base frequencies of 440 Hz (1 octave higher), 880 Hz (2 octaves higher), 1760 Hz (3 octaves higher), etc.

12-Note System. In standard Western Music practice, there are 12 distinct notes[1], each a semitone apart, as illustrated in the keyboard layout in Fig.1. The three just intervals discussed above are assigned as such: a Perfect Fifth is 7 semitones apart (C-G or E♭-B♭), a Major Third is 4 semitones apart (C-E or E♭-G), and an Octave is 12 semitones apart (C-C or E♭-E♭).

Pythagorean Tuning

Used in many Ancient Greek music, this tuning system is widely attributed to the famed Mathematician and Philosopher, Pythagoras.

Construction. In Pythagorean Tuning, the just intervals of the Octave (2:1) and the Perfect Fifth (3:2) are preserved, meaning that every note is constructed from these two intervals. Since a Perfect Fifth is 7 semitones apart, and an Octave is 12 semitones apart, and the numbers 7 and 12 are coprime, we can derive all 12 notes by stacking Perfect Fifths together. The construction process of Pythagorean tuning can be illustrated as:

[1] Notably, the Pythagorean tuning was widely attributed to Pythagoras (c. 570 BC – c. 495 BC), who died before the earliest written record that established the concept of having 12 distinct notes (*Elementa Harmonica* by Aristoxenus). Nonetheless, in this paper we will mainly discuss the performance of each tuning system for today's music, so for the purpose of this paper, we will assume that having 12 distinct notes is a well-established fact.

1. Assign a frequency to a base note;
2. Stack the 3:2 ratio to get new pitches that are a Perfect Fifth apart;
3. Use the 2:1 ratio to bring the pitches to different Octaves;
4. Repeat the process 6 times above and 5 times below.

While deriving frequencies for new pitches, we multiply the ratios while deriving higher pitches, and divide by the ratios for lower pitches. For example, if we start with the note A, we multiply the frequency of 3:2 to derive the pitch that is a Perfect Fifth above, E; and we divide by the frequency of 3:2 to derive the pitch a Perfect Fifth below, D.

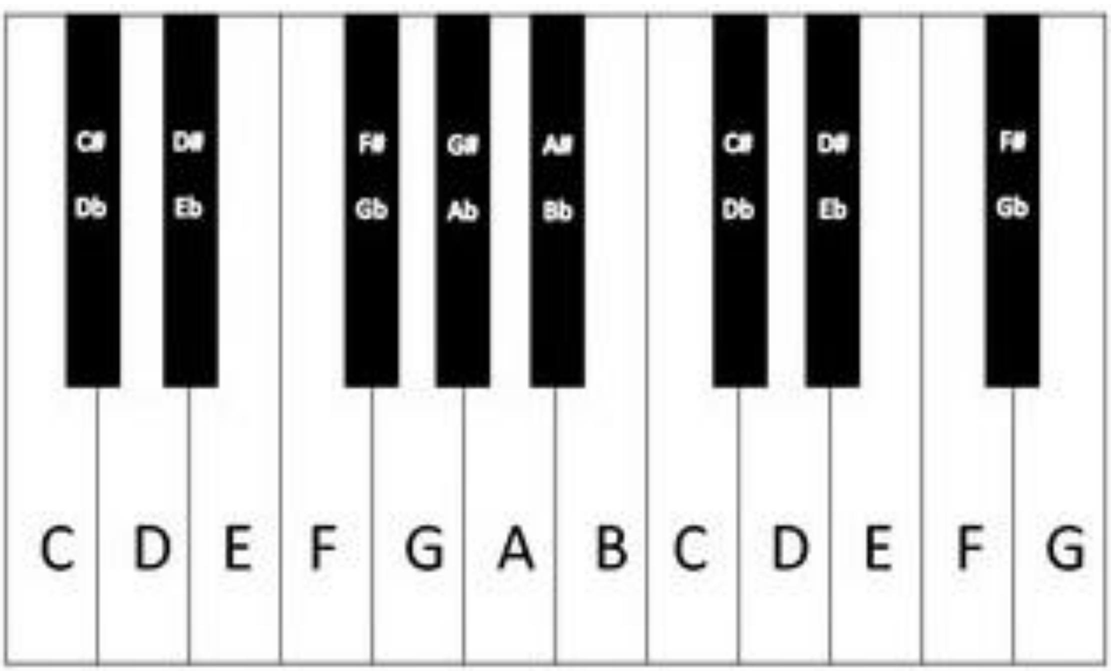

Fig. 1 Keyboard Layout

If we start with D as our base note and through these steps, we can obtain the following table:

Name	E♭	B♭	F	C	G	D	A	E	B	F♯	C♯	G♯
Formula	$\left(\frac{2}{3}\right)^5 \times 2^3$	$\left(\frac{2}{3}\right)^4 \times 2^3$	$\left(\frac{2}{3}\right)^3 \times 2^2$	$\left(\frac{2}{3}\right)^2 \times 2^2$	$\frac{2}{3} \times 2$	1	$\frac{3}{2}$	$\left(\frac{3}{2}\right)^2 \times \frac{1}{2}$	$\left(\frac{3}{2}\right)^3 \times \frac{1}{2}$	$\left(\frac{3}{2}\right)^4 \times \frac{1}{2^2}$	$\left(\frac{3}{2}\right)^5 \times \frac{1}{2^2}$	$\left(\frac{3}{2}\right)^6 \times \frac{1}{2^3}$
Ratio	$\frac{256}{243}$	$\frac{128}{81}$	$\frac{32}{27}$	$\frac{16}{9}$	$\frac{4}{3}$	1	$\frac{3}{2}$	$\frac{9}{8}$	$\frac{27}{16}$	$\frac{81}{64}$	$\frac{243}{128}$	$\frac{729}{512}$

Table 1 Pythagorean Tuning Table

Then after rearrangement, we will get:

Name	D	E♭	E	F	F♯	G	G♯	A	B♭	B	C	C♯
Ratio	1	$\frac{256}{243}$	$\frac{9}{8}$	$\frac{32}{27}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{729}{512}$	$\frac{3}{2}$	$\frac{128}{81}$	$\frac{27}{16}$	$\frac{16}{9}$	$\frac{243}{128}$

Table 2 Pythagorean Tuning Table in Order

Advantages: The stacking process is simple and intuitive. Also, all ratios we have obtained are rational, i.e. all just intervals. This is consistent with Pythagoras' philosophy that only rational numbers exist.

Disadvantages: While deriving the frequencies, we stopped at the note E♭. However, if we step down one more Perfect Fifth, we can calculate the ratio of A♭ as:

$$\frac{256}{243} \times \frac{2}{3} \times 2 = \frac{1024}{729} = 1.4047$$

whereas the G♯ we calculated has a ratio of

$$\frac{729}{512} = 1.4238$$

which means that G♯ and A♭ are two different notes, despite in the 12-note system, they are enharmonic equivalents, i.e. they occupy the same key in the keyboard.

In fact, the interval between G♯ and E♭ is not a Perfect Fifth. The ratio of this interval can be calculated as

$$\frac{256}{243} \times 2 : \frac{729}{512} = \frac{262144}{177147} = 1.4798$$

whereas a just Perfect Fifth has a ratio of $\frac{2}{3} = 1.5$.

This interval is formally called the Pythagorean Diminished Sixth (due to note spelling, G♯ and E♭ should theoretically be a sixth apart) but is more commonly called the Sour Fifth. It is significantly flatter than the just Perfect Fifth, and is perceived as quite dissonant. Therefore, although very elegant, the Pythagorean tuning have some severe disadvantages.

Quarter Comma Meantone Temperament

In the sixteenth and seventeenth century, the most common tuning system is Quarter Comma Meantone Temperament. This temperament was used by some of the most well-known composers such as Bach, Mozart and Beethoven. For brevity, this will be referred to simply as the "Meantone Temperament" in the following.

Construction. In Quarter Comma Meantone Temperament, the just intervals of the Octave (2:1) and the Major Third (5:4) are preserved, meaning that every note is constructed from these two intervals. Since a Perfect Fifth is 4 semitones apart, and an Octave is 12 semitones apart, and 4 divides 12, we cannot simply stack major thirds to get all 12 notes. Instead, the music theorists first derive the ratio of a Meantone Perfect Fifth by the following relation:

1. Stack 2 Octaves with a Major Third to get $12 \times 2 + 4 = 28$ semitones;
2. Stack 4 Perfect Fifths to get $7 \times 4 = 28$ semitones;
3. Calculate the Meantone Perfect Fifth by:

$$x^4 = 2^2 \cdot \frac{5}{4} \;\Rightarrow\; x = \sqrt[4]{5}$$

Then we can derive all 12 notes by a similar stacking process as Pythagorean Tuning, the difference being instead of using 3:2 as the ratio for Perfect Fifth, we are using $\sqrt[4]{5}$:1. Starting with D as the base note (the most common practice with Meantone Temperament), the resulted table would be:

Name	E♭	B♭	F	C	G	D	A	E	B	F♯	C♯	G♯
Formula	$\left(\frac{1}{\sqrt[4]{5}}\right)^5 \times 2^3$	$\left(\frac{1}{\sqrt[4]{5}}\right)^4 \times 2^3$	$\left(\frac{1}{\sqrt[4]{5}}\right)^3 \times 2^2$	$\left(\frac{1}{\sqrt[4]{5}}\right)^2 \times 2^2$	$\frac{1}{\sqrt[4]{5}} \times 2$	1	$\sqrt[4]{5}$	$(\sqrt[4]{5})^2 \times \frac{1}{2}$	$(\sqrt[4]{5})^3 \times \frac{1}{2}$	$(\sqrt[4]{5})^4 \times \frac{1}{2^2}$	$(\sqrt[4]{5})^5 \times \frac{1}{2^2}$	$(\sqrt[4]{5})^6 \times \frac{1}{2^3}$
Ratio	$\frac{8}{5\sqrt[4]{5}}$	$\frac{8}{5}$	$\frac{4}{5^{3/4}}$	$\frac{4}{\sqrt{5}}$	$\frac{2}{\sqrt[4]{5}}$	1	$\sqrt[4]{5}$	$\frac{\sqrt{5}}{2}$	$\frac{5^{3/4}}{2}$	$\frac{5}{4}$	$\frac{5\sqrt[4]{5}}{4}$	$\frac{5\sqrt{5}}{8}$

Table 3 Meantone Temperament Table

Rearrange to get:

Name	D	E♭	E	F	F♯	G	G♯	A	B♭	B	C	C♯
Ratio	1	$\frac{8}{5\sqrt[4]{5}}$	$\frac{\sqrt{5}}{2}$	$\frac{4}{5^{3/4}}$	$\frac{5}{4}$	$\frac{2}{\sqrt[4]{5}}$	$\frac{5\sqrt{5}}{8}$	$\sqrt[4]{5}$	$\frac{8}{5}$	$\frac{5^{3/4}}{2}$	$\frac{4}{\sqrt{5}}$	$\frac{5\sqrt[4]{5}}{4}$

Table 4 Meantone Temperament Table in Order

Advantages: The Meantone Temperament preserves the just Major Third, which is a sound typically associated with warmth. Therefore, musicians who play with this temperament can utilize this interval to convey a very warm feeling.

Disadvantages: Like in Pythagorean Tuning, if we step down one more Meantone Perfect Fifth from E♭, we can calculate the ratio of A♭ as:

$$\frac{8}{5\sqrt[4]{5}} \times \frac{1}{\sqrt[4]{5}} \times 2 = \frac{16}{5\sqrt{5}} = 1.4311$$

which is also inconsistent with the $\frac{5\sqrt{5}}{8} = 1.3975$ we calculated for G♯ above, making the enharmonic equivalents still inconsistent.

The interval between G♯ and E♭ can be calculated as:

$$\frac{8}{5\sqrt[4]{5}} \times 2 : \frac{5\sqrt{5}}{8} = \frac{128}{25 \times 5^{3/4}} = 1.5312$$

whereas a just Perfect Fifth has a ratio of $\frac{2}{3} = 1.5$, and a Meantone Perfect Fifth has a ratio of $\sqrt[4]{5} = 1.4953$.

This interval is commonly referred to as a Wolf Fifth. The Wolf Fifth is significantly sharper than a just Perfect Fifth, whereas a Meantone Perfect Fifth is virtually indistinguishable from a just Perfect Fifth.

Moreover, if we calculate the ratio of each semitone, we can get two different intervals:

D-E♭	E♭-E	E-F	F-F♯	F♯-G	G-G♯	G♯-A	A-B♭	B♭-B	B-C	C-C♯	C♯-D
1.0700	1.0449	1.0700	1.0449	1.0700	1.0449	1.0700	1.0700	1.0449	1.0700	1.0449	1.0700
D	C	D	C	D	C	D	D	C	D	C	D

Table 5 Meantone Temperament Semitone Ratios

The larger interval is called a "diatonic semitone" whereas the smaller interval is called a "chromatic semitone." Here the diatonic semitone is repeated at C♯-D-E♭ and G♯-A-B♭, two asymmetric positions. This means that sequence of semitone intervals is unique for each key, making each key sound different.

In his 1785 book *Ideas Towards an Aesthetic of Music*, Christian Friedrich Daniel Schubart dedicated 4 pages to describe the characterization of each musical key [3]. For example, when describing the color of B major, he wrote:

"B major, strongly colored, announcing wild passions, made up of the crudest colors. Anger, rage, jealousy, fury, desperation, and every burden of the heart lies in its sphere."

While discussing the keys, he skipped over the key of F♯ Major. This is because the Meantone Temperament has rendered this key practically unusable [4], for there are many clashing intervals in the key. Therefore, Meantone Temperament also has some major disadvantages.

12 Tone Equal Temperament

12 Tone Equal Temperament is the most common tuning system of today. It is the default tuning system of every tuner, every musical instrument, and every musical software.

Construction: Instead of trying to preserve just intervals, 12 Tone Equal Temperament keeps the same ratio for all 12 semitones. Therefore, the ratio of each semitone can be calculated by:

$$x^{12} = 2 \Rightarrow x = \sqrt[12]{2}$$

therefore, we get the following tuning table:

A	A♯/B♭	B	C	C♯/D♭	D	D♯/E♭	E	F	F♯/G♭	G	G♯/A♭
1	$\sqrt[12]{2}$	$2^{2/12}$	$2^{3/12}$	$2^{4/12}$	$2^{5/12}$	$2^{6/12}$	$2^{7/12}$	$2^{8/12}$	$2^{9/12}$	$2^{10/12}$	$2^{11/12}$

Table 6 12 Tone Equal Temperament Table

Advantage: The main achievement of 12 Tone Equal Temperament is consistency. First, we now have the consistency between enharmonic equivalents. Now, G♯ and A♭ are the same note, which reflects how they take up the same key on a keyboard. In addition, we have the consistency between intervals. If two intervals include the same number of semitones, they will sound the same, and there should not be any Sour Fifth or Wolf Fifth. Finally, the keys are also consistent in 12 Tone Equal Temperament. Because all the semitones are the same interval, each sequence of semitone yields the same result.

Disadvantage: By taking the 12th root of 2, we lost any just interval other than the Octaves. Because any non-octave-based interval is irrational, although in theory there are infinitely many just intervals, only the Octaves are truly “in tune” in 12 Tone Equal Temperament. Secondly, we have lost the characterization of the keys. Although the keys are inconsistent in the Meantone Temperament, each key is unique and flavorful, and composers were able to choose the key that best conveys their emotion. In today's tuning system, the subtlety is simply lost, and every key would sound the same.

Conclusion

The story of the tuning system is not a story about perfection. We did not arrive at a perfect tuning system; 12 Tone Equal Temperament has many flaws after all. Nor is it a story about degeneracy. The tuning system did not evolve to become worse, since we have solved many problems with 12 Tone Equal Temperament. Rather, it is a story about compromise. It is a story about how we gave up our obsession with the "purity" of mathematical forms or interval ratio to arrive at a tuning system that is compromised yet consistent.

References

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