# A Mathematical Study of Tessellations: The Art of Space Filling 

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## Introduction

The tessellation is the covering of a plane using geometric shapes without gaps or overlaps. First used by Sumerians in building walls decorations (Pickover, 2009), tessellations appear in various natural phenomenon and have been applied by artists and architects. In mathematics, researchers have regarded tessellations as a physical link between mathematical science and real-world life. In addition to the Euclidean space tessellations, mathematicians have generalized to higher dimensions and studied the internal abstract structures. In this paper, I will present the definition and categories of tessellations, the seventeen wallpaper groups related to the mathematical research of tessellations, and their connections with some previous topics covered in the Math 400 class throughout the whole semester.

## Definition and Categories

Mathematically, a tessellation is a partition of an infinite space into pieces having a finite number of distinct shapes. These geometric shapes are called tiles. In other words, the tiles cover an Euclidean or non-Euclidean plane with no gaps or overlaps. Based on the shapes of the tiles, there are three different categories of tessellations. A regular tessellation is made up of repetitive regular polygons of the same size and position. There are only three types of regular tessellations consisting of equilateral triangles, squares and regular hexagons, respectively. A vertex is defined as the point where the shapes join together. A semi-regular, or an Archimedean tessellation, is made up of more than one type of regular polygons in an isogonal arrangement. That is, The pattern at each vertex is identical (What is a tessellation?, 2018). There are right semi-regular tessellations, as are shown in Figure 1. The irregular tessellation has fewer
restrictions; it could be made up of any kind of geometric shapes, given that the shapes do not overlap or leave space.


Figure 1: Eight Semi-regular Tessellations

## Isometries of the Euclidean Plane and the Wallpaper Groups

When using a single tile to form a complete tessellation, the tile is always transformed in various ways, while some certain properties preserve. In an Euclidean plane, there are four ways of transforming the plane, called isometries of the Euclidean plane. These could be explain in. both mathematical and descriptive languages.

Mathematically, let $v$ be a vector in $R^{2}$, and let $p$ be a point in an Euclidean plane. The transformation, $T$, is a function such that $T(p)=p+v$. More visually, a transformation is to shift the plane in the direction $V$. Now let $\theta \in(0,2 \pi)$. Consider the $x-y$ coordinate plane, and let
$P(P x, P y)$ be an arbitrary point in the plane. A rotation around the origin, $R$, is denoted as $R(P)=\left(\begin{array}{cc}\cos \theta & -\sin \theta \\ \sin \theta & \cos \theta\end{array}\right)\left[\begin{array}{c}P x \\ P y\end{array}\right]$. A rotation around a point other than the origin could be accomplished by first translating the point to the origin, then performing the rotation. Let $c$ be another point of the plane and let $v$ be a unit vector in $R^{2}$. For an arbitrary plane $p$, the reflection, $F$, is denoted as $F(p)=p-2 t v$, where $t=(p-c) \cdot v$, the dot product of $(p-c)$ and $v$. Finally, the glide reflection is a combination of the transformation and the reflection. Let vector $w$ represents the direction the plane transforms and let $v$ represents the direction the plane reflects, the glide reflection, $G$, is denoted as $G(p)=w+F(p)$. Figure 2 gives a visualization of the four isometries of the Euclidean plane.


Figure 2: Isometries of the Euclidean Plane

Mathematicians have found that when using isometries to form a tessellation, there are only 17 different combinations. Each combination, called a wallpaper group, is formally defined as a type of topologically discrete group of isometries of the Euclidean plane that contains two linearly independent translations. For example, the group pl could be formed by single transformations, while the group $p 2$ shows a combination of translations and reflections.


Figure 3: (Left) Wallpaper Group p1; (Right) Wallpaper group p2

## Tessellations and Fractals

In Euclidean geometry, a fractal is a subset of Euclidean spaces such that all geometric figures or curves are self-similar. Fractals are frequently used in modeling structures in which similar patterns recur at smaller scales. Tessellations and fractals both possess the property of self-similarity; however, not all tessellations can form fractals. Specifically, such tessellations are called reptiles, shapes that can be dissected into smaller pieces of the same shape. A reptile fractal is formed by dissecting the reptile into smaller pieces, removing one or more copies of the subdivided shape, and repeating this process infinitely. The Sierpiński Carpet, named after Polish mathematician Wacław Sierpiński, is an example of reptile fractals. Figure 4 shows a complete process of constructing a Sierpiński Carpet. To begin, one starts with a square and cuts it into 9 identical pieces in a three by three grid. The central sub-square is then removed. This process is applied repeatedly to the remaining sub-squares, ad infinitum. Finally, one could claim that the area of the carpet is 0 . A proof of this claim could be reached using the concept of the limit:

Assume that the area of the original square if 1 . Then, after the first iteration, the area of the remaining part is $\frac{8}{9}$. Thus, the area of the carpet after the $i$ th iteration is $\left(\frac{8}{9}\right)^{i}$, with $\lim _{i \rightarrow \infty}\left(\frac{8}{9}\right)^{i}=0(\operatorname{arXiv}, 2012)$.


Figure 4: The Process Of Forming a Sierpiński Carpet

## Tessellations and the Golden Ratio

Mathematically, two quantities are in the golden ratio if the ratio of the larger to the smaller is equal to $\frac{1+\sqrt{5}}{2}$, represented as the Greek letter $\varnothing$. The golden ratio appears in both natural phenomenon and man-made art products. The Penrose tiling is closely connected with the golden ratio as well.

Named after English mathematician and physicist Roger Penrose who investigated it, a pair of the Penrose tiling consists of two kinds of triangles, which could be combined together to form either a group of a 'kite' and a 'dart' or a pair of a 'fat' rhombus and a 'thin' rhombus.

Figure 5 shows an example of such combinations.


Figure 5: (Left) A Kite and A Dart; (Right) Two Rhombuses


Figure 6: A Regular Pentagon

For each of the kite and the dart, the ratio of the long side to the short side is exactly the golden ratio, $\emptyset$. Accordingly, one could find a group of kite and darts in a regular pentagon, as is shown in Figure 6. In this case, the area of the 'kite' to that of the 'dart' is the golden ratio as well (How to make a Penrose tiling, n.d.). In addition, for each rhombus shown in Figure 5, the ratio of the long diagonal to the short diagonal is the golden ratio as well. If one repeat the two combinations in an infinite plane to create a tessellation, the ratio between the number of each type of tiles is used, both in the case of the kite and dart combination and the rhombuses, approaches the golden ratio, $\varnothing$. That is,

$$
\lim _{\text {Number of tiles } \rightarrow \infty} \frac{\text { Number of kites }}{\text { Number of darts }}=\lim _{\text {Number of tiles } \rightarrow \infty} \frac{\text { Number of 'fat' rhombuses }}{\text { Number of 'thin' rhombuses }}=\infty
$$

This is because of one unique property of tessellations: the placement of one type of tiles over another depends only on the geometric shapes of the tiles themselves instead of the tessellation designer (Schultz). Additionally, one may tile each Penrose tile using smaller tiles of the same shape infinitely; this self-similarity shows that a Penrose tiling is a kind of fractals as well.

## Conclusion:

Tessellations have connect abstract mathematical studies of geometry with visualized representations that could be found in everyday life. While certain tessellations may look complex and chaotic, some internal structures and characteristics always possess. This paper gives a brief introduction to how mathematics is used to describe, to investigate and to generate
tessellations. For future studies, the technique of computer science and image processing could be employed in designing decorative tessellations as well as studying more complex tessellations in a mathematical context.

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