Mathematics in Gomoku

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Introduction

Gomoku, also called Five in a Row, is a popular abstract strategy board game originating from Japan. The game is also popular in China where it is called Wuziqi(五子棋). It is a special case of an m,n,k-games(with m=15, n=15 ,k=5) which are mainly of mathematical interest. In this paper, I will introduce the basic rules of Gomoku, the tournament opening rules and relate some mathematical aspects of it to graph theory and Ramsey games.

Basic Rules

An m,n,k-game is also called a k-in-a-row game on an m-by-n board in which two players take turns in placing a stone of their color on an m-by-n board, the winner being the player who first gets k stones of their own color in a row, horizontally, vertically, or diagonally. Thus, tic-tac-toe is the 3,3,3-game and gomoku is the 15,15,5-game. In Gomoku(15,15,5-game), players alternate turns placing a stone of their color on an empty intersection. Black plays first. The winner is the first player to form an unbroken chain of five stones horizontally, vertically, or diagonally. The winning cases are presented in Figure 1.

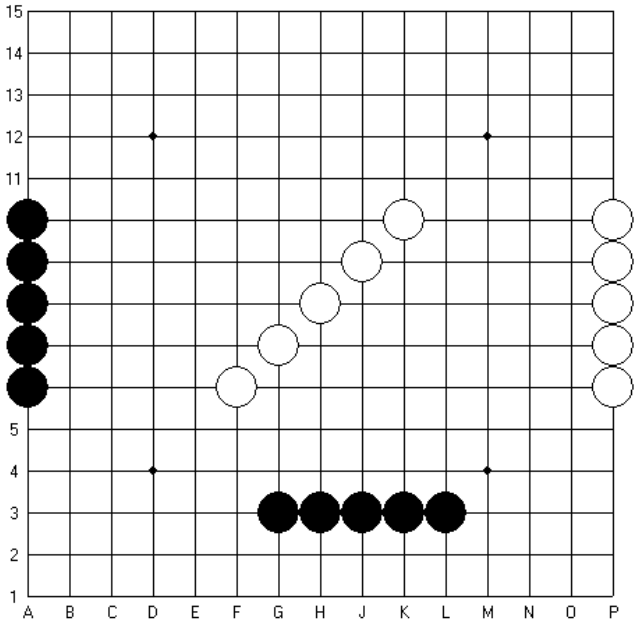

Figure1: winning cases of Gomoku [1]

Winning Strategy

The first-player (Black) has a winning strategy for Gomoku. We can prove this using the argument of strategy stealing.The strategy stealing argument assumes that the second player actually has a winning strategy. The first player makes an arbitrary move. After that, the player pretends that she is the second player and adopts the second player's winning strategy. She can do this as long as the strategy does not call for placing a stone on the 'arbitrary' position that is already occupied. But this extra stone can only help her, she can again play an arbitrary move and continue as before with the second player's winning strategy. This ensures that the first player wins, contradicting the assumption that the second player has a winning strategy. This ensures that the first player wins, contradicting the assumption that the second player has a winning strategy.

After people proved that the Black player has a winning strategy, people still did not find a way to solve the game until computers and artificial intelligence became common tools for mathematicians. People used different computer programs to solve Gomoku. For instance, in 1975, Knuth and Moore used the α-β search algorithm to develop a winning strategy for the Black player[2]. After that, in 1993, Ailis, Herik and Huntjens solved Gomoku by using Threat-space search, which was a more efficient method and the CPU could respond within 0.1 second[3].

Tournament Opening Rules

Since the player who controls the black pieces has advantages, in order to make the game more balanced, in the professional tournament, certain moves are forbidden. For example, in Renju, a professional variant of gomoku, there are certain moves that Black is not allowed to make:[4]

- Double three – Black cannot place a stone that builds two separate lines with three black stones in unbroken rows (i.e. rows not blocked by white stones).
- Double four – Black cannot place a stone that builds two separate lines with four black stones in a row.
- Overline – six or more black stones in a row.

There are also other rules like Pro, Long Pro and Swap. Since 2009, the tournament rule used for the Gomoku world championships is the Swap2 opening rule.Three stones, two black and one white, are placed on the board by the tentative first player. After then, the potential second player has three options:

1. They can choose to play as black
2. They can choose to play as white and place a second white stone
3. Alternatively, they can place two more stones, one black and one white, and let the tentative first player choose which color to play.[5]

Because the first player has no idea where the second player will place the extra stones if they

choose option 2 or 3, the swap2 opening protocol prevents one of the players from studying a line excessively.

All these rules are used to eliminate or minimize the advantages of the first player and ensure it is a fair game.

Ramsey Game

The Ramsey game is a simpler game than Gomoku, which could be considered as an ancestor of Gomoku. The rules are simple. Two players start with a complete graph, which is a set of points of all which are connected by lines. Then the two players take turns coloring one of the edges. The winner is the first person to color edges which form a triangle. For example, figure 2 is an example with a complete graph on 5 points.

First, we need enough vertices in order to have a winner. Ramsey theory states that there exists a least positive integer R(r, s) for which every blue-red edge coloring of the complete graph on R(r, s) vertices contains a blue clique on r vertices or a red clique on s vertices. (Here R(r, s) signifies an integer that depends on both r and s.)[6] Notice that it is trivial when n = 3 or n = 4 since we can easily find a case where no player forms a triangle. Also, as shown in the following figure3, we can see that it is possible that neither of two players forms a triangle when n = 5.

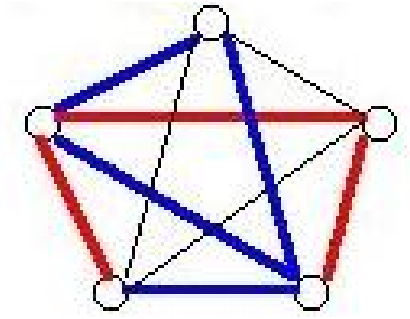

Figure2

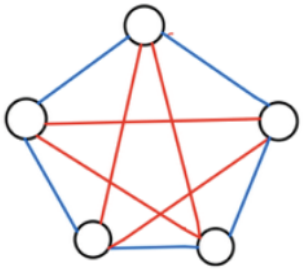

Figure3

Now consider the case where the graph has 6 vertices. Ramsey proved that R(3, 3) = 6 and any 2-coloring of K6 yields a triangle. Thus, in this case, there must be a winner no matter how bad each player's strategy is. Similar to Gomoku, Ramsey Game also ensures that the first player has natural advantages. We can prove that using the same strategy stealing argument we discussed previously in the winning strategy for Gomoku section. Besides the strategy stealing

approach, I will provide a step-by-step proof of how the first player will definitely win the game. Suppose that the first player will color the edges with blue and the second player color the edges with red, as shown in Figure 4. Now the first player can make his or her first two moves from the same vertex so that the second player's first move will not be the third edge of the triangle. Then the second player must color the third edge of the triangle since otherwise, the first player wins by coloring the third edge of the triangle.In this case, the first player now can draw another edge from the same vertex so that it guarantees that there are two choices to form a triangle on the next move. The second player cannot make 2 consecutive moves so he or she cannot block both of them, as shown in the last part of Figure 4. The second player cannot make 2 consecutive moves so he or she cannot block both of them.

This similar to the double three in the gomoku we talked about in the Opening Rules. In the gomoku, the other player cannot block both of the threes at the same time, which is shown in figure 5.

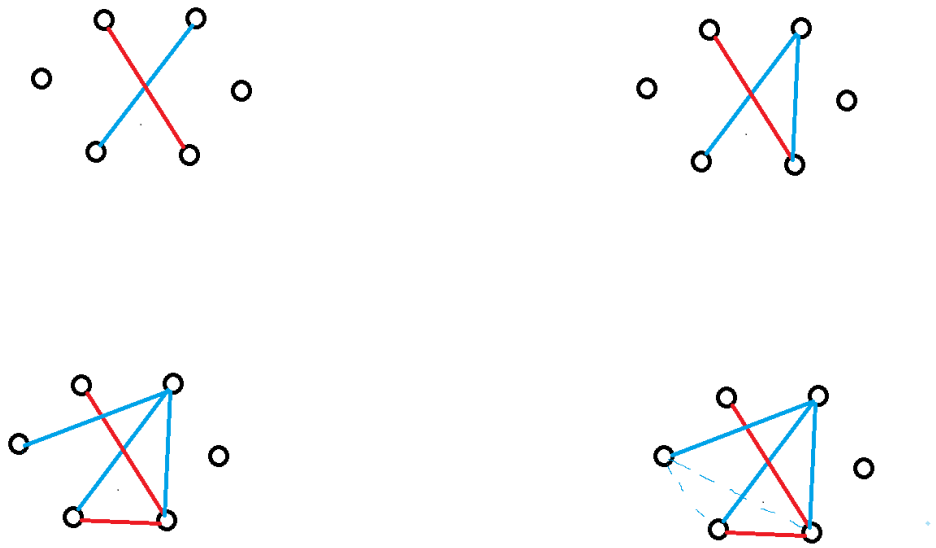

Figure 4

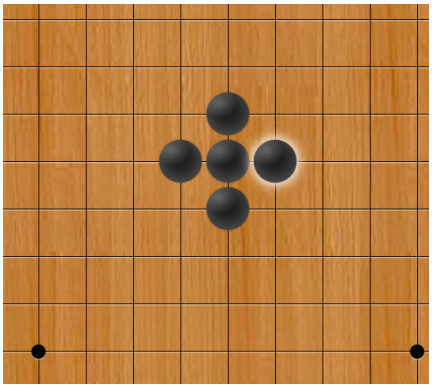

Figure 5

Conclusion

Although there are similarities between Gomoku and Ramsey Game, I acknowledge that it is hard for mathematicians to solve Gomoku by treating it as a Ramsey game since Ramsey Game requires two players to play on a complete graph while the 15x15 board of gomoku cannot be considered as a complete graph. Unfortunately, using AI to solve Gomoku seems to be the only approach right now since computers can examine thousands of possible moves of the opponent in less than a second. However, this paper does not focus on finding a mathematical solution of Gomoku, but instead, it makes an effort to connect mathematical concepts and board games like Gomoku. My intention is to introduce Gomoku to new people from a mathematical perspective and hope people could discover other interesting mathematical aspects in Gomoku or even in more complex board games like the game of GO and chess.

References

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