

Photography from Mathematical Perspectives

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Abstract

Photography as an art form arose from advancements in technology that allow diverse manipulations for desired artistic expression. From basic parameters to aesthetic intuition, photography combines practical knowledge from multiple disciplines such as mathematics, physics, and chemistry. This paper discusses the mathematics behind the fundamentals of photography, starting from basic parameters to the construction of an aesthetically pleasing picture. The three basic parameters in photography are aperture, shutter speed, and ISO, with geometric sequences buried behind their formation to enable quick calculation. The scales of three parameters together define the exposure value. Yet different parametric combinations under equivalent exposure can construct opposing visual presentation of the same object. Photographic compositions are the arrangements photographers use to guide the viewer's eye towards the most important elements. Out of many general guidelines established on geometry and psychology, this paper focuses on the famous rule of thirds which mimics the golden ratio. The Fibonacci sequence is the mathematics behind the formation of golden ratio and its Binet's formula is proven with the help of a lemma. This paper also adds a section to briefly introduce computational photography as recommended by comments.

I. Introduction

Photography as an art form arose from advancements in technology that allow diverse manipulations for desired artistic expression. Photographers can drastically change the outcome of an image through choosing various cameras, lenses, film, framing and timing of a shot. Filters, studio lighting, various darkroom processes, and digital enhancement add even more tools for photographers to manipulate their images. Therefore, it is not surprised to know that photography, especially modern digital photography, involves quite a bit behind-the-scene mathematics.

The behind-the-scene mathematics includes geometric sequences, polyhedrons, fractions, geometry, and so on. It is these mathematical principles that support the convenient functioning of cameras. But mathematics does not only exist in physical operations, but also in aesthetical compositions and interpretations. For example, using geometry when analyzing graphic composition can quickly decode the protagonist and moving trend in a photo.

II. Basic Parameters

i. Aperture

Aperture, first of the three pillars of photography, is defined as the opening in a lens through which light passes to enter the camera. Aperture affects not only exposure but also depth of field.

Its basic mechanism imitates how human's eyes work in different light settings: as human moves between bright and dark environments, the iris in human's eyes either expands or shrinks to control the size of the pupil. For the camera system, aperture is the pupil, determining how much amount of light can pass through. Physically, blades in lens form a diaphragm to block any light that tries to pass, except through the center.

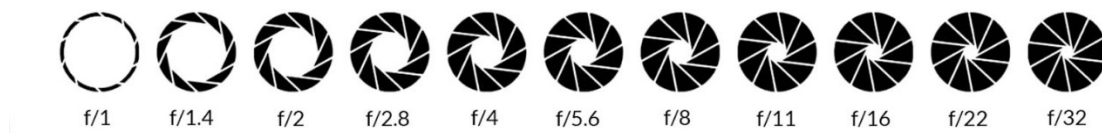


Figure 1. Common aperture sizes

To numerically describe different sizes of aperture, f-stop is utilized for notation. For f-stop, also called f-number, small numbers represent large apertures, whereas large numbers represent small apertures. The explanation for this counter-intuitive design is a simple fact that apertures are fractions. When photographers talk about a f-stop of f/16, for example, it is referring to the fraction one-sixteenth. Here the letter “f” stands for “focal length”. When we substitute focal length into the fraction, we are solving for the diameter of the aperture. For example, we now have an 80-200mm f/2.8 lens fully zooms out to 80mm. If our f-stop is set to f/4, the diameter of the aperture blades in your lens will look exactly 20 millimeters across (80mm divided by 4), whereas at f/16, the diameter will be reduced to merely 5 millimeters (80mm divided by 16).

The common f-stops consist not of successive integers. But after giving a second look, we can tell that the scale approximates a geometric sequence: the sequence of the powers of the square root of 2.

Fraction	Exact number	F-stop
$\sqrt{2}^0$	1	f/1.0
$\sqrt{2}^1$	1.41421356237	f/1.4
$\sqrt{2}^2$	2	f/2.0
$\sqrt{2}^3$	2.82842712475	f/2.8
$\sqrt{2}^4$	4	f/4.0
$\sqrt{2}^5$	5.65685424949	f/5.6
$\sqrt{2}^6$	8	f/8.0
$\sqrt{2}^7$	11.313708499	f/11

Table 1. Corresponding fraction to common f-stop

Such approximation is not a coincidence but a deliberate choice. When the aperture diameter increases by the square root of two, the open area of the aperture is doubled. As a result, the amount of light that gets in is doubled. The reason for choosing the “ $\sqrt{2}$ ” scale, then, is that it keeps the intervals equivalent to the doubling and halving of exposures. This is extremely helpful when photographers have to calculate the exposure value by hand. The f-stop scale does not operate exclusively with whole numbers. But most professional cameras still keep the photographic convention of operating only with thirds or halves of a stop besides whole numbers.

Below is the corresponding table of f-stops, diameter of aperture, radius of aperture, and area of aperture to compare data more directly. Here we assume the focal length is 50mm. The open area of aperture is calculated by the equation:

$$S(fl, fs) = \pi \times \left(\frac{\text{Foca length}}{F_{\text{stop}}} \right) / 2)^2 \quad (eq.1)$$

F-stop	Diameter of aperture (mm)	Radius of aperture (mm)	Area of aperture (sq. mm)
shown on lens	50mm divided by f-stop	Half of the diameter	π times the squared radius
f/1.0	50.0	25.0	1963
f/1.4	35.7	17.9	1002
f/2.0	25.0	12.5	491
f/2.8	17.9	8.9	250
f/4.0	12.5	6.3	123
f/5.6	8.9	4.5	63
f/8.0	6.3	3.1	31
f/11	4.5	2.3	16
f/16	3.1	1.6	8
f/22	2.3	1.1	4

Table 2. Corresponding area of aperture to common f-stop

ii. Shutter speed

A camera takes a photograph by exposing a digital sensor or a piece of film to light. Shutter is a barrier that keeps light out when the camera is not taking photos, while shutter speed is how long that barrier stays open to let light into the image. Shutter speed varies

from milliseconds to minutes, sometimes even hours. Similar to aperture, shutter speed also represents fraction. Smaller the fraction, faster the shutter opens and closes.

The main reason to adjust shutter speed is to achieve proper image brightness. In the following collection of images, shutter speed increases from left to right. As a result, image appears brighter with longer shutter speed.



Figure 2. How image brightness changes with shutter speed [1]

Meanwhile, motion blur effects should not be overlooked: shorten or lengthen shutter speed can be used for creating freezing action or blurring motion. In the example below, the left one is taken using quicker shutter speed to create a snapshot, while the right one is taken using longer shutter speed to create silky water flow effect.



Figure 3. How shutter speed affects motion blur [2]

The standard shutter speeds are a set of discrete numbers, also forming a geometric sequence: 1, 1/2, 1/4, 1/8, 1/15, 1/30, 1/60, 1/125, 1/250, 1/500, 1/1000, 1/2000, 1/5000. Moving from 1/5000 second to 1 second, each increase in shutter speed increases the time and the amount of light hitting the film by a factor of two. Notice the jumping from 1/60 to 1/125, instead of 1/60 to 1/120, is deliberately designed so that the doubling pattern is simpler for later half of the sequence. The irregular gap between 1/8 and 1/15 is of the same reason. Therefore, the entire standard shutter speeds actually consist of three separate

geometric sequences, with a multiplying factor of 1/2 in each of them. Such sequence is chosen for convenient calculation of exposure similar to the f-stop scale.

iii. ISO

ISO originally stands for International Organization for Standardization. But ever since two film standards called ASA and DIN were combined into ISO standards in 1974, “ISO” refers to the film standard in common sense [3].

ISO initially indicates how sensitive a film was to light. Lower the ISO, lower the film sensitivity, finer the grain in the product picture. It was later adopted by digital camera manufacturers for the purpose of maintaining similar brightness levels as film. Instead of describing it as image sensor sensitivity, ISO is like a mapping to tell your camera how bright the output photo should be, given a particular input exposure.

Common ISO values used in most cameras are ISO 100, 200, 400, 800, 1600, 3200, 6400, 12800. Similar to the Standard shutter speed, we can easily recognize that it is a geometric sequence designed for convenient calculation. It is calculated by the following equation with k denoting the gear number:

$$ISO(k) = 100 \times 2^{k-1} \quad (eq.2)$$

With all else equal, when photographers double the ISO, they double the brightness of the photo. For instance, a photo set at ISO 400 will be twice brighter than at ISO 200 with other equal settings.

The cost of choosing higher ISO is that the image will have more grain or noise the higher number photographers set ISO at. Therefore, a general recommendation is to stick to the base ISO, which is the lowest native ISO on the camera, to get the highest image quality.

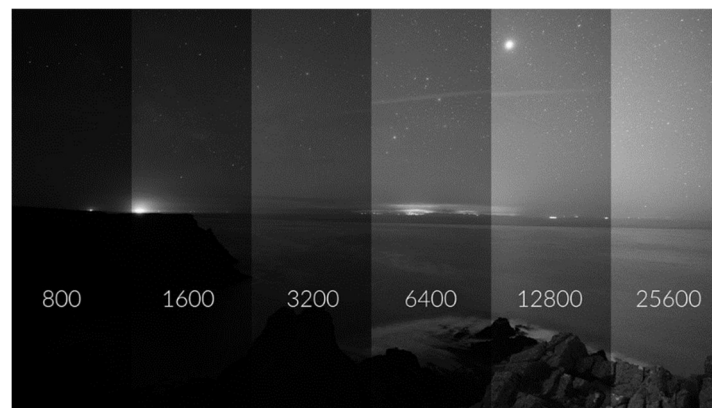


Figure 4. How ISO affects brightness [4]

III. Parameter Combinations

As mentioned above, all three parameters we discussed have the same main function of controlling exposure, just by different means. Therefore, fundamental parameters work together to determine the final exposure effect. Moreover, different combinations of these three parameters can construct different photographic effects under the same exposure. Knowing how to coordinate parameters is as crucial as understanding the function of each parameter.

i. Exposure

The simplified equation for calculating photographic exposure (EV) using three parameters is:

$$EV = \log_2\left(\frac{100 \times S(fl,fs)^2}{ISO(k) \times t}\right) \quad (eq.3) [5]$$

Notice that exposure often increases or decreases several orders of magnitude at a time. Therefore, usually we use log number to represent its value.

Using such construction, the complicated exposure calculation is simplified to quick equations using only small numbers. It is the regulation of fundamental parameters using geometric sequences that allows us to approximate and adjust the exposure quickly during a photoshoot.

Be aware that exposure value calculated using this equation can only represent the parameter combinations of the camera. It is not relevant to environment brightness. To match camera setting with light available in the environment, photographers will use exposure compensation.

ii. Equivalent exposure

As mentioned before, different combinations of these three parameters can construct different photographic effects under the same exposure. There are several benefits to use all three variables for controlling the final exposure.

First, having them all allows photographers to vary the light over a wider range. For example, the combination of 1/1000 second and f/16 lets in very little light, while 1 second and f/1.4 lets in a large amount of light. If we can control only shutter speed or f-stops, we will not reach such a wide range.

Second, different combinations of those three parameters can create same amount of exposure with different artistic looks. That is tuning under equivalent exposure.

For shutter speed, if photographers want to freeze the moment, a fast shutter speed of 1/1000 can be their choice. Conversely, a slow shutter speed of 1 second can be used to blur a fast-moving subject for creating momentums in the picture.



Figure 5. Shadows of a soul VI - Disruption [6]

As briefly mentioned before, aperture affects how much of the scene is in focus, that is, the depth of field. It is an important concept of use to address the protagonist or motif. Larger the aperture, deeper the depth of field. For example, if we use an $f/16$ f-stop to take a photograph of a man standing in front of a tree, both him and the tree will be in focus. If instead we use an $f/2$ f-stop, only the person will be in focus. Photographers choose to blur the background or not depending on where they want the focus of this photo to be.

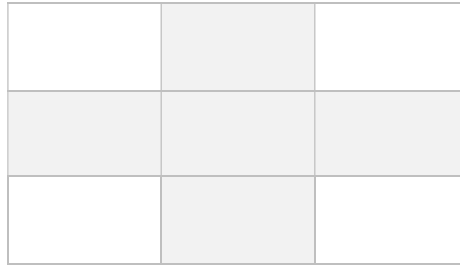
IV. Photographic Composition

While appropriate parameter combinations help to create desired motions, photographic compositions allow photographers to structure the picture visually. That is, the framing and arrangement we can use to create certain visual effect such as emphasizing motif by contrast.

There are various photographic composition guidelines based on geometry and human visual habits. For example, lines are commonly used to create certain mood and elicit emotion in photographers' work. Horizontal lines typically tend to portray a sense of stability and consistency. Using horizontal lines, for example, photographers can convey a feeling of rest or the message that there is a lack of change in a scene. Objects that appear horizontally in a photograph can serve as a dividing line in the composition or provide an anchor to the picture's subject. Vertical lines, on the other hand, help to give interpretation to the mood. These lines elicit powerful emotions and convey strength, often providing a sense of length or height. The photos below use both horizontal and vertical lines, creating a contrast between stability and strength.

In general, there are five basic compositional rules to help provoke a stronger emotional response in the viewer: leading lines, rule of thirds, genitive space. Horizon line, and symmetry and patterns [7]. The following discussion will focus on one of the most profound guidelines: the rule of third.

i. Rule of thirds



Imagine the image is divided into nine sections evenly like the grid above. The rule of thirds suggests photographers to place their subject or point of interest on one of the intersections of those lines or along one of the lines.

Rule of thirds draws the viewer's eye into the composition, instead of letting them glance through the center. By placing the subject off center, photographers also embrace more blank space, highlighting an interesting background. Meanwhile, off-centered subjects tend to convey more of a feeling of motion than centered ones.

Looking at the mathematics behind the construction, rule of third is an imprecise imitation of the golden ratio or Fibonacci sequence. The golden ratio, also referred to as the golden rectangle or golden mean, is a shape with a proportion of 1 to 1.618. The exact origins of the theory are unknown, but there are mentions of the golden ratio everywhere.

The major reason behind the wide usage and acceptance of golden ratio is that it mimics the look of nature which creates a sense of balance and harmony. The mystery behind its mechanism is still not fully studied. There're many hypotheses for why we are attracted to patterns following golden ratio. One leading hypothesis is that patterns with golden ratio are very common in nature and human gradually embraced such visual representation with positive feelings. It might be a mutual reinforcement where the prevalence of golden ratio in nature motivates more usage which in turn enhance people's acceptance of golden ratio. This may relate to the concept of mere exposure effect in psychology: people tend to develop a preference for things merely because they are familiar with them. When this is repeated for thousands of years, the implicit link between golden ratio patterns and beauty is naturally strengthened.

In common photography, the images can be composed using it in various orientations. It can be turned clockwise or not. The mirror image of the spiral works as well as the spiral itself.

ii. Fibonacci Sequence

The mathematic theory of Fibonacci sequence dates more than 750 years and was developed by an Italian mathematician nicknamed "Fibonacci". He also helped spread the use of the current number system through Europe during his lifetime. Its connection to golden ratio is that if we put together any two successive numbers in the Fibonacci sequence, the result ratio will approximately be 1 to 1.618. Larger the numbers get, closer the ratio will be to the golden ratio.

In the Fibonacci sequence, each number is the sum of the two preceding ones, starting from 0 and 1.

$$F_n = F_{n-1} + F_{n-2} \quad (\text{for } n > 1) \quad (\text{eq.4}) [8]$$

Since the Fibonacci sequence is defined by a linear recurrence with constant coefficients, it has a closed-form solution known as “Binet’s formula” where Φ denotes the golden ratio and F_n denotes the n th term of the Fibonacci sequence. It was derived by Binet in 1843, although the result was known to Euler, Daniel Bernoulli, and de Moivre More than a century earlier.

$$\begin{aligned} F_n &= \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}} \\ &= \frac{\Phi^n - (-\Phi)^{-n}}{\sqrt{5}} \end{aligned} \quad (\text{eq.5}) [9]$$

As n approaches infinity, the ratio of the Fibonacci terms approaches the Golden Ratio. To express it mathematically, we have

$$F_n \approx \left(\frac{1+\sqrt{5}}{2}\right) F_{n-1} \quad (\text{eq.6})$$

To prove Binet’s formula, we must first derive a lemma. The proof below follows one from Ross Honsberger’s *Mathematical Gems* [10].

Lemma I:

For any solution x of $x^2 - x - 1 = 0$, $x^n = xF_n + F_{n-1}$, $n \geq 1$.

Proof of Lemma I:

For $n = 0$ and $n = 1$, $F_0 = 0$ and $F_1 = 1$. Therefore $x^1 = xF_1 + F_0 = x$.

Assume for some n , $x^n = xF_n + F_{n-1}$. We want to prove that $x^{n+1} = xF_{n+1} + F_n$. First, we present x^{n+1} as the multiplication of x^n and x .

$$\begin{aligned} x^{n+1} &= x \cdot x^n \\ &= x \cdot (xF_n + F_{n-1}) \\ &= x^2 \cdot F_n + x \cdot F_{n-1} \end{aligned}$$

Since $x^2 - x - 1 = 0$, we have $x^2 = x + 1$.

$$\begin{aligned} x^{n+1} &= (x + 1) \cdot F_n + x \cdot F_{n-1} \\ &= x(F_n + F_{n-1}) + F_n \\ &= xF_{n+1} + F_n \end{aligned} \quad (\text{by eq.4})$$

Therefore, Lemma I is proved by induction.

Proof of Binet’s formula:

Since the golden ratio denoted by Φ and $-\frac{1}{\Phi}$ are solutions of $x^2 - x - 1 = 0$, we can apply the Lemma I to them and conclude that $\Phi^n = \Phi F_n + F_{n-1}$ and $(-\Phi^{-1})^n = -\Phi^{-1}F_n + F_{n-1}$.

$$\begin{aligned}\text{Thus, } \Phi^n - (-\Phi^{-1})^n &= \Phi F_n + F_{n-1} + \Phi^{-1} F_n - F_{n-1} \\ &= (\Phi + \Phi^{-1}) F_n.\end{aligned}$$

$$\text{Therefore, we have } F_n = \frac{\Phi^n - (-\Phi^{-1})^n}{\Phi + \Phi^{-1}}.$$

$$\text{Since } \Phi + \Phi^{-1} = \frac{1+\sqrt{5}}{2} + \frac{2}{1+\sqrt{5}} = \frac{1+\sqrt{5}-1+\sqrt{5}}{2} = \sqrt{5},$$

$$F_n = \frac{\Phi^n - (-\Phi)^{-n}}{\sqrt{5}} \quad (\text{eq.5})$$

V. Advancement in Photography

Photography as a subject has changed substantially mainly due to all the technical improvements but also the public's aesthetics shifts. The following section discusses only two examples of photographic advancements, while the progress of technical improvement keeps rolling forward and scholar's opinion on such advancements alters over time.

i. Computational photography

Though the exact definition of computational photography is still under debate, one of its key features is to use processing power to overcome the limitations of camera's conventional hardware. The concept of computational photography was put forward fairly early, while the incorporation of it into camera-related hardware was not popularized until recent years due to limiting processing power back in the old days. For example, we can largely reduce noise in your photo by combining multiple images. While we have to do it manually back then, the cameras nowadays can do (and actually are doing) such edition for us in real time.

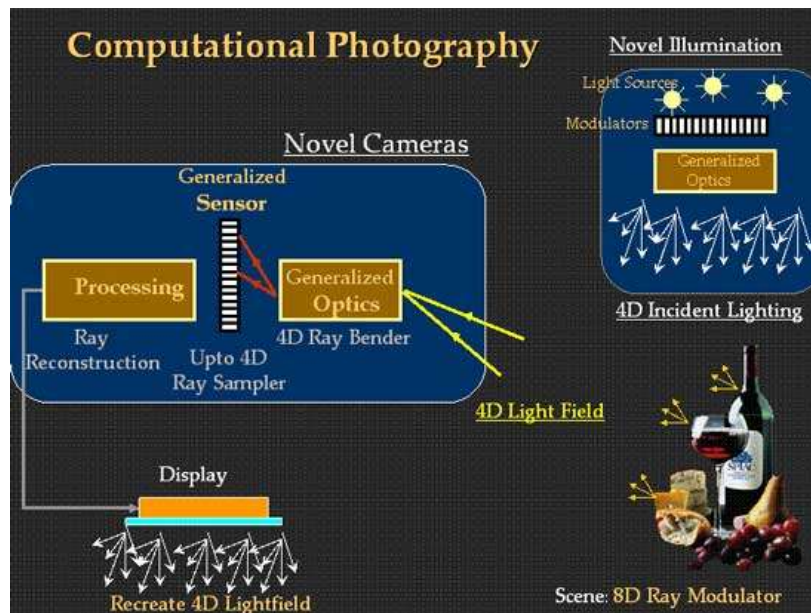


Figure 6. Shadows of a soul VI - Disruption [11]

Instead of optical processes, computational photography uses digital computation to capture and edit photo quickly. Thus, its research places at the intersection of computer graphics, computer vision, and human-computer interaction. Computational photography tends to record multi-dimensional information on multiple layers usually by altering conventional hardware, allowing refocus or 3D imaging later during edit.

ii. Group f.64

The technical advancement of photography evoked diverse responses from the public, the scholars, and especially the artists. Since emerged in late 1860s, the Pictorialism made its way among West Coast academic artists and became the most popular photographic perspective around the first few decades of the nineteenth century. Instead of documenting the reality, the Pictorialist perspective utilize the camera as an artistic tool to emphasize the beauty of subject matter, tonality, and composition.

Group f.64 was formed in 1932 against the prevalence of the Pictorialism. As revealed by its name, this group of California photographers usually adopts the smallest setting of a large-format camera diaphragm aperture for a particularly good resolution and depth of field. They stand for the unmanipulated “pure” documentation, i.e., using the camera to record life as it is [12].

Group f.64’s revolt against pictorialism seems particularly important as we discuss the essence of photography especially facing its technical advancements: with a wider range of possible manipulations, what principle should we adopt to regulate photography? Should we embrace all approaches, or should we set restrictions on certain excessive technical alternations? What artistic meaning does photography have with layers of technical improvement, if any? The broader question behind such inquiry is how humanity should react and respond to the improvement of technology, not only artistically but also under the humanity-technology relation framework.

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