Notes on Advanced Linear Algebra

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1 Complex vectors and complex matrices

In applications and theoretical development, it is important to study complex vectors and matrices. Let $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ be the set of natural numbers, integers, rational numbers, real numbers, complex numbers, respectively.

1.1 Complex numbers: Basic operations

- A complex number has the standard form $z = a + ib$ with $a, b \in \mathbb{R}$, and we have the complex plane representation. The complex conjugate of z is $\bar{z} = a - ib$.

- For $z_1, z_2 \in \mathbb{C}$, one can perform addition $z_1 + z_2$, subtraction $z_1 - z_2$, multiplication $z_1 z_2$, and division z_1/z_2 provided $z_2 \neq 0$.

- The size, modulus, or norm of $z = a + ib$ is $|z| = \sqrt{a^2 + b^2}$, the argument of z is $\theta \in [0, 2\pi)$ or $\mathbb{R}$ with $\cos\theta = a/|z|$ and $\sin\theta = b/|z|$. Note that $z\bar{z} = \bar{z}z = |z|^2$.

- The polar form of z is $z = |z|(\cos\theta + i\sin\theta) = |z|e^{i\theta}$. If $z_1 = |z_1|e^{i\theta_1}$ and $z_2 = |z_2|e^{i\theta_2}$, then $z_1 z_2 = |z_1||z_2|e^{i(\theta_1+\theta_2)}$, where we may replace $\theta_1 + \theta_2$ by $\theta_1 + \theta_2 - 2\pi$ in case $\theta_1 + \theta_2 \geq 2\pi$. If $z_2 \neq 0$, then $z_1/z_2 = (|z_1|/|z_2|)e^{i(\theta_1-\theta_2)}$, where we may replace $\theta_1 - \theta_2$ by $\theta_1 - \theta_2 + 2\pi$ in case $\theta_1 < \theta_2$.

1.2 Real or Complex Vectors and Matrices

Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and $\mathbb{F}^n$ be the set of column vectors with n co-ordinates.

- If $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{F}^n$, and $\gamma \in \mathbb{R}$, then the addition and scalar multiplication are defined by
$$\mathbf{x} + \mathbf{y} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix} \quad \text{and} \quad \gamma\mathbf{x} = \begin{pmatrix} \gamma x_1 \\ \vdots \\ \gamma x_n \end{pmatrix},$$
respectively.

- The set $\mathbb{F}^n$ form a vector space under addition and scalar multiplication.

  The addition is closed, associative, commutative; there is a zero vector $\mathbf{0} \in \mathbb{F}^n$ such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$; for any $\mathbf{x} \in \mathbb{F}^n$ there is an additive inverse $-\mathbf{x}$ such that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$; the scalar multiplication always yields an element in $\mathbb{C}^n$ and satisfies $\gamma_1(\gamma_2\mathbf{x}) = (\gamma_1\gamma_2)\mathbf{x}$ and $1\mathbf{x} = \mathbf{x}$ for any $\gamma_1, \gamma_2 \in \mathbb{F}$ and $\mathbf{x} \in \mathbb{F}^n$.

Let $M_n(\mathbb{F})$, $M_{m,n}(\mathbb{F})$ be the set of $n \times n$ and $m \times n$ matrices over $\mathbb{F}$, respectively. We write $M_n, M_{m,n}$ if $\mathbb{F} = \mathbb{C}$.

- If $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \in M_{m+n}(\mathbb{F})$ with $A_1 \in M_m(\mathbb{F})$ and $A_2 \in M_n(\mathbb{F})$, we write $A = A_1 \oplus A_2$.

- x^t, A^t denote the transpose of a vector x and a matrix A.

- For a complex matrix A, $\overline{A}$ denotes the matrix obtained from A by replacing each entry by its complex conjugate. Furthermore, $A^* = (\overline{A})^t$.

- If $A = (a_{ij})$ is $m \times n$, and $B = (b_{jk})$ is $n \times p$, then $C = AB = (c_{ik})$ is $m \times p$ such that $c_{ik} = a_{i1}b_{1k} + \cdots + a_{in}b_{nk}$ for $1 \le i \le m, 1 \le k \le p$.

- If $A = (A_{ij})$ is such that A_{ij} is $m_i \times n_j$ for $1 \le i \le r, 1 \le j \le s$, and $B = (B_{jk})$ is such that B_{jk} is $n_j \times p_k$ for $1 \le j \le s$ and $1 \le k \le q$, then $C = AB = (C_{ik})$ such that $C_{ik} = A_{i1}B_{1k} + \cdots + A_{is}B_{sk}$ for $1 \le i \le r, 1 \le k \le q$.

- If $A \in M_{mn}$ has columns $\mathbf{u}_1, \ldots, \mathbf{u}_n$ and $B \in M_{n,p}$ has rows $\mathbf{v}_1^t, \ldots, \mathbf{v}_n^t$, then
$$AB = \sum_{j=1}^{n} \mathbf{u}_i \mathbf{v}_j^t.$$

1.3 Basic concepts and operations for complex vectors & matrices

We can extend the concepts on real vectors and real matrices to complex vectors and complex matrices.

- Linear equations, solution sets, elementary row operations.

  Example. Consider $Ax = b$ with $A = \begin{pmatrix} 1 & 3i \\ 2-i & h \end{pmatrix}$, $b = (1, i)^t$. Consider h such that the system is solvable.

- Column space, row space, null space, and rank of a complex matrix.

  Determine h in the above example so that A has rank one or rank 2. Also, determine bases for the column space, row space, and null space of A for each choice of h.

- Determinant, eigenvalues, eigenvectors, diagonal form.

  Compute the determinant of A above. Find the eigenvalues, eigenvectors of A if $h = 1$.

- To solve for eigenvalues and eigenvectors,

  1) Solve the characteristic equation $\det(\lambda I - A) = 0$ to find the eigenvalues.

  Note that $\det(\lambda I - A) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_n)$ by the Fundamental Theorem of Algebra.

  2) For each root λ_i of $\det(\lambda I - A) = 0$, find a basis for solution set of $(\lambda_i I - A)x = 0$.

  3) There are n linearly independent eigenvectors $x_1, \ldots, x_n$ corresponding the $\lambda_1, \ldots, \lambda_n$ if and only if $AS = SD$, where S has columns $x_1, \ldots, x_n$, and $D = \text{diag}\,(\lambda_1, \ldots, \lambda_n)$, so that $S^{-1}AS = D$. We say that A is diaonalizable.

  Note that if A has n distinct eigenvalues, then A is diagonalizable because each eigenvalue has at least one eigenvector, and these eigenvectors are linearly independent.

  For example, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is not diagonalizable.

- Vector spaces, basis, change of bases.

  The space $\mathbb{C}^n$ has dimension n, a linearly indpendent set (or a spanning set) $\{v_1, \ldots, v_n\}$ of n vectors form a basis. This happen if and only if the matrix S with column $v_1, \ldots, v_n$ is invertible, equivalently, $\det(S) \neq 0$.

- Linear transformations, range space, kernel.

  A matrix $A \in M_{m,n}$ define a linear transformation $T : \mathbb{C}^n \to \mathbb{C}^m$ such that $T(x) = Ax$ for any $x \in \mathbb{C}^n$. The column space of A is the range space, the null space of A is the kernel.

1.4 Inner product, orthonormal sets, Gram-Schmidt process

Recall that the inner product of $u, v \in \mathbb{C}^n$ is $\langle u, v\rangle = v^*u$ and satisfies the following:

(1) For any $u, u_1, u_2, v \in \mathbb{C}^n$ and $a, b \in \mathbb{C}$, $\langle au_1 + bu_2, v\rangle = a\langle u_1, v\rangle + b\langle u_2, v\rangle$

(2) For any $u, v \in \mathbb{C}^n$, $\langle u, v\rangle = \overline{\langle v, u\rangle}$

(3) For any $u \in \mathbb{C}^n$, $\langle u, u\rangle \geq 0$, the equality holds if and only if $u = 0$.

The Euclidean norm (a.k.a. ℓ_2-norm) of $v \in \mathbb{C}^n$ is defined by $\|v\| = (v^*v)^{1/2}$ and satisfies the following.

(a) For any $v \in \mathbb{C}^n$, $\|v\| \geq 0$. (positive definiteness)
The equality holds if and only if $v = 0$.

(b) For any $a \in \mathbb{C}$ and $v \in \mathbb{C}$, $\|av\| = |a|\|v\|$. (absolute homogeneity)

(c) For any $u, v \in \mathbb{C}^n$, $\|u + v\| \leq \|u\| + \|v\|$. (triangle inequality)
The equality holds if and only if one vector is a nonnegative multiple of the other.

Condition (c) follows from

(d) $|\langle u, v\rangle| \leq \|u\|\|v\|$. (Cauchy-Schwartz inequality)
The equality holds if and only if one vector is a multiple of the other.

A set of vectors $\{u_1, \dots, u_m\} \subseteq \mathbb{F}^n$ is orthonormal if $\langle u_i, u_j\rangle = \delta_{ij}$, the Kronecker delta such that $\delta_{jj} = 1$ and $\delta_{ij} = 0$ if $i \neq j$. Equivalently, $U^*U = I_m$, where $U \in M_{n,m}(\mathbb{F})$ has columns $u_1, \dots, u_m$.

Note: An orthonormal set $\{u_1, \dots, u_m\} \subseteq \mathbb{F}^n$ is always linearly independent so that $m \leq n$. A vector v is a linear combination of $u_1, \dots, u_m$ if and only if $v = a_1u_1 + \cdots + a_mu_m$ with

$$a_j = \langle v, u_j\rangle \text{ for } j = 1, \dots, m.$$

Gram-Schmidt Process Let $v_1, \dots, v_m \in \mathbb{F}^n$ be linearly independent with $m < n$.
Set $u_1 = v_1/\|v_1\|$.
For $k > 1$, let $f_k = v_k - (a_1u_1 + \cdots + a_{k-1}u_{k-1})$ with $a_j = u_j^*v_k$ and $u_k = f_k/\|f_k\|$.

Then $\{u_1, \dots, u_k\}$ is an orthonormal basis for span$\{v_1, \dots, v_k\}$ for $k = 1, \dots, m$.
If $m < n$, one may further extend $\{u_1, \dots, u_m\}$ to an orthonormal basis $\{u_1, \dots, u_n\}$.
To see this, one can apply the Gram-Schmidt process to the basic columns of the rank n matrix $[u_1 \cdots u_m \; e_1 \cdots e_n]$.

A set $\{u_1, \dots, u_n\}$ is an orthonormal basis for $\mathbb{F}^n$ if and only if the matrix U with columns $u_1, \dots, u_n$ satisfies $U^*U = I_n$. When $\mathbb{F} = \mathbb{C}$, the matrix U is called a unitary matrix; when $\mathbb{F} = \mathbb{R}$, the matrix U is called an orthogonal matrix.

We will denote by $U_n(\mathbb{F})$ the set of matrices $U \in M_n(\mathbb{F})$ such that $U^*U = I_n$.

Exercises

1. Let $A = \begin{pmatrix} 1 & 2i & 3 & 4 \\ 2i & 6 & 1+i & 1-i \\ 1+2i & 6+2i & 4+i & 5-i \end{pmatrix}$.

   (a) Reduce the matrix to row echelon form, and find the rank of A.

   (b) Find bases for the row space, column space, and null space of A.

   (c) Solve the equations $Ax = (2, 2-i, 3-i))^t$ and $Ax = (1, 0, 0)^t$.

2. Let $A = \begin{pmatrix} i & 2 \\ -2 & i \end{pmatrix}$.

   (a) Find the eigenvalues λ_1, λ_2 of A, and the corresponding unit eigenvectors u_1, u_2.

   (b) Let $U = [u_1 \ u_2]$. Show that $U^*U = I_2$ and $AU = UD$ with $D = \operatorname{diag}(\lambda_1, \lambda_2)$.

   (c) Show that $A^k = UD^kU^* = \lambda_1^k v_1 v_1^* + \lambda_2^k v_2 v_2^*$ for all (positive or negative) integers k

3. Suppose $A = SDS^{-1} \in M_n$ such that $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$, and where S has columns $x_1, \dots, x_n$ and S^{-1} has rows $y_1^t, \dots, y_n^t$.

   (a) Show that $y_i^t x_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$ [Hint: Consider $S^{-1}S$.]

   (b) Show that $A^k = SD^kS^{-1} = \sum_{j=1}^n \lambda_j^k x_j y_j^t$ for every positive integer k.

   (c) If A is invertible, show that $A^k = SD^kS^{-1} = \sum_{j=1}^n \lambda_j^k x_j y_j^t$ for every negative integer k.

   (d) For any polynomial $f(z) = a_m z^m + \cdots + a_0$, let $f(A) = a_m A^m + \cdots + a_1 A + a_0 I_n$. Show that $f(A) = \sum_{j=1}^n f(\lambda_j) x_j y_j^t$.

4. Suppose $A = \begin{pmatrix} 1 & i \\ 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} i & 0 & 0 \\ 2 & 2i & 0 \\ 1 & 1 & 3i \end{pmatrix}$.

   (a) Show that for any $C \in M_{2,3}$, there is $X \in M_{2,3}$ such that $AX + C = XB$.

   [Hint: Let $X = [x_{ij}]$ and set up a linear system of 6 equations to solve for $[x_{ij}]$ for a given C.]

   (b) Suppose $T = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ for some matrix $C \in M_{2,3}$. Show that there is $X \in M_{2,3}$ such that

   $$TS = S(A \oplus B) \text{ if } S = \begin{pmatrix} I_2 & X \\ 0 & I_3 \end{pmatrix}. \text{ Find } S^{-1} \text{ and conclude that } S^{-1}TS = A \oplus B.$$

   (c) Show that conclusion (a) may fail if A and B share a common eigenvalue.

5. Let $u, v_1, v_2 \in \mathbb{C}^n, a, b \in \mathbb{C}$. Show that $\langle u, av_1 + bv_2\rangle = \bar{a}\langle u, v_1\rangle + \bar{b}\langle u, v_2\rangle$.

6. Let $S = \{v_1, \ldots, v_k\} \subseteq \mathbb{C}^n$ be an orthonormal set. Show that S is linearly independent.

7. Let $u, v \in \mathbb{C}^n$. Prove the Cauchy-Schwarz inequality $|\langle u, v\rangle| \le \|u\|\|v\|$, and the triangle inequality $\|u + v\| \le \|u\| + \|v\|$, and determine the conditions for equality.

   Hint: Let $u, v \in \mathbb{C}^n$ be nonzero. Consider $e^{i\theta}$ such that $\langle u, e^{i\theta}v\rangle = |\langle u, v\rangle|$ so that $\langle e^{i\theta}v, u\rangle = \bar{\langle} u, e^{i\theta}v\rangle = |\langle u, v\rangle|$. Then for any $t \in \mathbb{R}$,

$$0 \le \|u + te^{i\theta}v\|^2 = at^2 + 2bt + c$$

   with $a = \|v\|^2, c = \|u\|^2, b = |\langle u, v\rangle|$. Then argue that $b^2 \le ac$ to prove the inequality, and argue that the equality hold if and only if $u + te^{i\theta}v = 0$ for some $t \in \mathbb{R}$.

8. Let $v_1 = (1, i, 1)^t, v_2 = (1, i, 2)^t$.

   (a) Apply Gram-Schmidt process to the vectors v_1, v_2 to get an orthonormal pairs u_1, u_2.

   (b) Let $A = [u_1\ u_2]$. Solve the system $A^*x = (0, 0)^t$.

   (c) Determine u_3 such that $\{u_1, u_2, u_3\}$ is an orthonormal basis for $\mathbb{C}^3$.

9. Let $u = (1, 2i, 1 - i)^t$. Find a unitary U with $u/\|u\|$ as the first column.

10. Suppose $A \in M_{n,m}$ with $m \le n$ with rank m. Show that $A = PU$ such that $P \in M_{n,m}$ has orthonormal columns, and U is upper triangular.

11. Let $A = \begin{pmatrix} 1 & 1-i & 2+i \\ 1 & 1+i & -2+i \\ i & i & 2 \end{pmatrix}$. Write $A = UR$ for an upper triangular matrix R.

    [Apply Gram Schmidt to the columns of A to get a unitary matrix U.]

2 Unitary equivalence and unitary similarity

Two matrices $A, B \in M_{m,n}$ are unitarily equivalent if there are unitary $U \in M_m$ and $V \in M_n$ such that $A = UBV$. Two matrices $X, Y \in M_n$ are unitarily similar if there is a unitary $W \in M_n$ such that $X = W^*YW$. It is easy to show that these are equivalence relations, that is, reflective, symmetric and transitive.

In this chapter, we consider different canonical forms of matrices under unitary equivalence and unitary similarity.

2.1 Singular value decomposition

Lemma 2.1.1 *Let A be a nonzero $m \times n$ matrix, and $u \in \mathbb{C}^m, v \in \mathbb{C}^n$ be unit vectors such that $|u^*Av|$ attains the maximum value. Suppose $U \in M_m$ and $V \in M_n$ are unitary matrices with u and v as the first columns, respectively. Then $U^*AV = \begin{pmatrix} u^*Av & 0 \\ 0 & A_1 \end{pmatrix}$.*

Proof. Note that the existence of the maximum $|u^*Av|$ follows from basic analysis result.

Suppose $U^*AV = (a_{ij})$. If the first column $x = U^*Av = (a_{11}, \ldots, a_{m1})^t$ has nonzero entries other than a_{11}, then $\tilde{u} = Ux/\|Ux\| = Ux/\|x\| \in \mathbb{C}^m$ is a unit vector such that

$$\tilde{u}^*Av = x^*U^*Av/\|x\| = x^*x/\|x\| = \|x\| > \sqrt{|a_{11}|^2} = |a_{11}| = |u^*Av|,$$

which contradicts the choice of u and v. Similarly, if the first row $y^* = x^*AV = (a_{11}, \ldots, a_{1n})$ has nonzero entries other than a_{11}, then $\tilde{v} = Vy/\|Vy\| = Vy/\|y\|$ is a unit vector satisfying

$$u^*A\tilde{v} = u^*AVy/\|y\| = y^*y/\|y\| = \|y\| > |a_{11}|^2,$$

which is a contradiction. The result follows. □

Theorem 2.1.2 *Let A be an $m \times n$ matrix of rank r. Then there are unitary matrices $U \in M_m, V \in M_n$ such that*

$$U^*AV = D = \sum_{j=1}^{r} s_j E_{jj}.$$

As a results, if U and V have columns $\mathbf{u}_1, \ldots, \mathbf{u}_m \in \mathbb{C}^m$ and $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{C}^n$,

$$A = \sum_{j=1}^{r} s_j \mathbf{u}_j \mathbf{v}_j^*$$

Proof. We prove the result by induction on $\max\{m, n\}$. By the previous lemma, there are unitary matrices $U \in M_m, V \in M_n$ such that $U^*AV = \begin{pmatrix} u^*Av & 0 \\ 0 & A_1 \end{pmatrix}$. We may replace U by

$e^{i\theta}U$ for a suitable $\theta \in [0, 2\pi)$ and assume that $u^*Av = |u^*Av| = s_1$. By induction assumption there are unitary matrices $U_1 \in M_{m-1}, V_1 \in M_{n-1}$ such that $U_1^*A_1V_1 = \begin{pmatrix} s_2 & & \\ & s_3 & \\ & & \ddots \end{pmatrix}$. Then $([1] \oplus U_1^*)U^*AV([1] \oplus V_1)$ has the asserted form, where r is the rank of A. □

Remark 2.1.3 The values $s_1 \geq \cdots \geq s_r > 0$ are the nonzero **singular values** of A, which are $s_1^2, \ldots, s_r^2$ are the nonzero eigenvalues of AA^* and A^*A. The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are the right singular vectors of A, and $\mathbf{u}_1, \ldots, \mathbf{u}_r$ are the left singular vectors of A. So, they are uniquely determined. We will denote the singular values of A by $s_1(A) \geq s_2(A) \geq \cdots$

Here is another way to do the singular value decomposition. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\} \subseteq \mathbb{C}^n$ be an orthonormal set of eigenvectors corresponding to the nonzero eigenvalues $s_1^2, \ldots, s_r^2$ of A^*A. Let $\mathbf{u}_j = A\mathbf{v}_j/s_j$. Then $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\} \subseteq \mathbb{C}^m$ is an orthonormal family such that $A = \sum_{j=1}^r s_j \mathbf{u}_j \mathbf{v}_j^*$.

Similarly, let $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\} \subseteq \mathbb{C}^m$ be an orthonormal set of eigenvectors corresponding to the nonzero eigenvalues $s_1^2, \ldots, s_r^2$ of AA^*. Let $v_j = A^*\mathbf{u}_j/s_j$. Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\} \subseteq \mathbb{C}^n$ is an orthonormal family such that $A = \sum_{j=1}^r s_j \mathbf{u}_j \mathbf{v}_j^*$.

If $A \in M_{m,n}$, then one can find real orthogonal matrices $U \in M_m$ and $V \in M_n$ with columns $\mathbf{u}_1, \ldots, \mathbf{u}_m$ and $\mathbf{v}_1, \ldots, \mathbf{v}_n$ such that $A = U(\sum_{j=1}^r s_j E_{jj})V^* = \sum_{j=1}^r s_j \mathbf{u}_j \mathbf{v}_j^*$.

We may extend the definition of inner product $\langle x, y \rangle$ and inner product norm $\|x\|$ for vectors $x, y \in \mathbb{F}^n$ to matrices by

$$\langle A, B \rangle = \sum_{i,j} a_{ij}\bar{b}_{ij} = \mathrm{tr}\,(AB^*) \quad \text{and} \quad \|A\|_F = \langle A, A \rangle^{1/2}$$

if $A = (a_{ij}), B = (b_{ij}) \in M_{m,n}$. $\|A\|_F$ is called the Frobenius norm or ℓ_2-norm of A.

Theorem 2.1.4 *Suppose $A \in M_{m,n}(\mathbb{F})$ has rank r and singular value decomposition $A = \sum_{j=1}^r s_j \mathbf{u}_j \mathbf{v}_j^*$, where $s_1 \geq \cdots \geq s_r > 0$ $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\} \subseteq \mathbb{F}^m, \{\mathbf{v}_1, \ldots, \mathbf{v}_r\} \subseteq \mathbb{F}^n$ are orthonormal sets. For any positive integer $k \leq r$, $A_k = \sum_{j=1}^k s_j \mathbf{u}_j \mathbf{v}_j^*$ satisfies*

$$\|A - A_k\|_F \leq \|A - X\|_F \qquad \text{for all } X \in M_{m,n} \text{ with rank at most } k.$$

If $k \geq r$, then no approximation is needed.

Proof. Let B has rank k such that $\|A - B\|_F$ is minimum among all B with rank at most k. Then there are unitary $P \in M_m$ and $Q \in M_n$ such that $PBQ = \sum_{j=1}^k b_j E_{jj}$ with $b_j = s_j(B)$ for $j = 1, \ldots, k$. Since $\|PXQ\|_F = \|X\|_F$, if $PAQ = (a_{ij})$, then

$$\|A - B\|_F^2 = \|P(A - B)Q\|_F^2 = \sum_{i \neq j} |a_{ij}|^2 + \sum_{j=1}^k |a_{jj} - b_j|^2 + \sum_{j > k} |a_{jj}|^2.$$

Let $C = P(A-B)Q = (c_{ij})$. If there is $1 \le i \le k$ such that $c_{ij} \ne 0$, we may change the (i,j) entry of PBQ to a_{ij} to get a rank at most r matrix $\hat{B}$ so that $\|A - \hat{B}\|_F$ is smaller. Similarly, if there is $1 \le j \le k$ such that such that $c_{ij} \ne 0$, we may change the (i,j) entry of PBQ to a_{ij} to get a rank at most k matrix $\hat{B}$ so that $\|A-\hat{B}\|_F$ is smaller. Hence, at the minimum, $P(A-B)Q = \begin{pmatrix} 0_r & 0 \\ 0 & A_{22} \end{pmatrix}$. So, $PAQ = \begin{pmatrix} \sum_{j=1}^k b_j E_{jj} & 0 \\ 0 & A_{22} \end{pmatrix}$, and $b_1, \dots, b_k$ are singular values of A. Thus,

$$\|PAQ - PBQ\|_F^2 = \mathrm{tr}\,(AA^*) - \sum_{j=1}^k b_j^2 = \sum_{j=1}^r s_j(A)^2 - \sum_{j=1}^k b_j^2,$$

which is minimum if $(b_1, \dots, b_r) = (s_1(A), \dots, s_k(A))$. □

Note that the A_k is uniquely determined if and only if $s_k(A) > s_{k+1}(A)$.

2.2 Schur Triangularization lemma and its consequences

Theorem 2.2.1 *Let $A \in M_n$ and $\det(\lambda I - A) = \prod_{j=1}^n (\lambda - \lambda_j)$. Then there is a unitary U such that U^*AU is in upper (or lower) triangular form with diagonal entries $\lambda_1, \dots, \lambda_n$.*

Proof. By induction on n. If $n = 1$, the results holds. Assume the results holds for matrices of sizes smaller than n, and $A \in M_n$. Let $Ax = \lambda_1 u_1$ for a unit vector u_1, and U is unitary with first column of U_1 equal to u_1. Then $U_1^* A U_1 = \begin{pmatrix} \lambda_1 & * \\ 0 & A_2 \end{pmatrix}$. By induction assumption, there is $V_1 \in M_{n-1}$ such that $V_1^* A_2 V_1 = T$ is in triangular form. If $U = U_1([1] \oplus V_1)$, then $U^*AU = \begin{pmatrix} \lambda_1 & * \\ 0 & V^*A_2V \end{pmatrix} = \begin{pmatrix} \lambda_1 & * \\ 0 & T \end{pmatrix}$ is in upper triangular form. □

Note that $\lambda_1, \dots, \lambda_n$ can be arranged in any order we like. Some of the λ_j could be the same. If $\mu_1, \dots, \mu_r$ are distinct and $\det(\lambda I - A) = \prod_{j=1}^r (\lambda - \mu_j)^{m_j}$, we say that A has distinct eigenvalues $\mu_1, \dots, \mu_r$ with multiplicities $m_1, \dots, m_r$, respectively.

Theorem 2.2.2 (Cayley-Hamilton) *Let $A \in M_n$ and $f(\lambda) = \det(\lambda I - A) = \sum_{j=0}^n a_j \lambda^j$. Then*

$$f(A) = \sum_{j=0}^n a_j A^j = 0_n.$$

Proof. We need to show that $\sum_{j=0}^n a_j A_j = (A - \lambda_1 I) \cdots (A - \lambda_n I_n) = 0$. It suffices to show that

$$0 = Z = [U^*(A - \lambda_1 I)U] \cdots [U^*(A - \lambda_n I_n)U],$$

where $U^*AU = (a_{ij})$ is in upper triangular form with diagonal entries $\lambda_1, \dots, \lambda_n$. Then $B_j = U^*(A - \lambda_j I)U$ is in upper triangular form with (j,j) entry equal to zero. .

We will prove by induction on n that if $B_1, \dots, B_n \in M_n$ are matrices in upper triangular form, and the (j,j) entry of B_j equals zero for $j = 1, \dots, n$, then $B_1 \cdots B_n = 0_n$.

For $n = 1$, the result is trivial. For $n = 2$, the product B_1 and B_2 has the form

$$\begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix} \begin{pmatrix} * & * \\ 0 & * \end{pmatrix},$$

which is clearly equal to 0_2.

Suppose the result holds for matrices in M_{n-1}. Let $B_j = \begin{pmatrix} * & * \\ 0 & T_j \end{pmatrix}$ for $j = 1, \dots, n$. Then by block multiplication of $B_2 \cdots B_n$, and the induction assumption on $T_2 \cdots T_n = 0_{n-1}$, we have

$$B_1 \cdots B_n = \begin{pmatrix} 0 & * \\ 0 & T_1 \end{pmatrix} \begin{pmatrix} * & * \\ 0 & T_2 \cdots T_n \end{pmatrix} = \begin{pmatrix} 0 & * \\ 0 & 0_{n-2} \end{pmatrix} = 0_n.$$

Now, let $B_j = U^*(A - \lambda_j I)U = U^* A_j U - \lambda_j I$. We get the desired result. □

Remark People have the misconception that $\det(\lambda I - A) = 0$ is valid if we put $\lambda = A$ in the above equation so that $\det(\lambda I - A) = \det(A - A) = \det(0_n) = 0$. In the theorem, we actually put $x^k = A^k$ in $f(x) = a_0 + \cdots + x^n$ and conclude that $f(A) = 0_n$, the zero matrix.

2.3 Normal matrices

Definition 2.3.1 *(1) A matrix $A \in M_n$ is normal if $AA^* = A^*A$. (2) A matrix $A \in M_n$ is Hermitian if $A = A^*$. (3) A matrix $A \in M_n$ is positive semidefinite if $\mathbf{x}^* A \mathbf{x} \ge 0$ for all $x \in \mathbb{C}^n$. (4) A matrix $A \in M_n$ the matrix A is positive definite if $\mathbf{x}^* A \mathbf{x} > 0$ for all nonzero $x \in \mathbb{C}^n$. (5) A matrix $A \in M_n$ is unitary $A^*A = I_n$.*

Theorem 2.3.2 *A matrix $A \in M_n$ is normal if and only if $A = UDU^*$ for a diagonal matrix D, i.e., A is unitarily diaogonalizable.*

Proof. If $U^*AU = D$, i.e., $A = UDU^*$ for some unitary $U \in M_n$. Then $AA^* = UDU^*UD^*U = UDD^*U^* = UD^*DU^* = UD^*U^*UDU^* = A^*A$.

Conversely, suppose $U^*AU = (a_{ij}) = \tilde{A}$ is in upper triangular form. If $AA^* = A^*A$, then $\tilde{A}\tilde{A}^* = \tilde{A}^*\tilde{A}$ so that the $(1,1)$ entries of the matrices on both sides are the same. Thus,

$$|a_{11}|^2 + \cdots + |a_{1n}|^2 = |a_{11}|^2$$

implying that $\tilde{A} = [a_{11}] \oplus A_1$, where $A_1 \in M_{n-1}$ is in upper triangular form. Now,

$$[|a_{11}|^2] \oplus A_1 A_1^* = \tilde{A}\tilde{A}^* = \tilde{A}^*\tilde{A} = [|a_{11}|^2] \oplus A_1^* A_1.$$

Consider the $(1,1)$ entries of $A_1A_1^*$ and $A_1^*A_1$, we see that all the off-diagonal entries in the second row of A_1 are zero. Repeating this process, we see that $\tilde{A} = \text{diag}\,(a_{11}, \ldots, a_{nn})$. $\square$

Proposition 2.3.3 *A matrix $A \in M_n$ is unitary if and only if it is unitarily similar to a diagonal matrix with all eigenvalues having modulus 1.*

Proof. If $U^*AU = D = \text{diag}\,(\lambda_1, \ldots, \lambda_n)$ with $|\lambda_1| = \cdots = |\lambda_n| = 1$, then A is unitary because

$$AA^* = UDU^*UD^*U^* = U(DD^*)U^* = UU^* = I_n.$$

Conversely, if $AA^* = A^*A = I_n$, then $U^*AU = D = \text{diag}\,(\lambda_1, \ldots, \lambda_n)$ for some unitary $U \in M_n$. Thus, $I = U^*IU = U^*AUU^*A^*U = DD^*$. Thus, $|\lambda_1| = \cdots = |\lambda_n| = 1$. $\square$

Theorem 2.3.4 *Let $A \in M_n$. The following are equivalent.*

(a) *A is Hermitian.*

(b) *A is unitarily similar to a real diagonal matrix.*

(c) *$\mathbf{x}^*A\mathbf{x} \in \mathbb{R}$ for all $\mathbf{x} \in \mathbb{C}^n$.*

Proof. Suppose (a) holds. Then $AA^* = A^2 = A^*A$ so that $U^*AU = D = \text{diag}\,(\lambda_1, \ldots, \lambda_n)$ for some unitary $U \in M_n$. Now, $D = U^*AU = U^*A^*U = (U^*AU)^* = D^*$. So, $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Thus (b) holds.

Suppose (b) holds and $A = U^*DU$ such that U is unitary and $D = \text{diag}\,(d_1, \ldots, d_n$. Then for any $\mathbf{x} \in \mathbb{C}^n$, we can set $U\mathbf{x} = (y_1, \ldots, y_n)^t$ so that $\mathbf{x}^*A\mathbf{x} = \mathbf{x}^*U^*DU\mathbf{x} = \sum_{j=1}^n d_j|y_j|^2 \in \mathbb{R}$.

Suppose (c) holds. Let $A = H + iG$ with $H = (A + A^*)/2 \geq 0$ and $G = (A - A^*)/(2i)$. Then $H = H^*$ and $G = G^*$. Then for any $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{x}^*H\mathbf{x} = \mu_1 \in \mathbb{R}, \mathbf{x}^*G\mathbf{x} = \mu_2 \in \mathbb{R}$ so that $\mathbf{x}^*A\mathbf{x} = \mu_1 + i\mu_2 \in \mathbb{C}$. If G is nonzero, then $V^*GV = \text{diag}\,(\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \neq 0$. Suppose $\mathbf{x}$ is the first column of V. Then $\mathbf{x}^*A\mathbf{x} = \mathbf{x}^*H\mathbf{x} + i\mathbf{x}^*G\mathbf{x} = \mu_1 + i\lambda_1 \notin \mathbb{R}$, which is a contradiction. So, we have $G = 0$ and $A = H$ is Hermitian. $\square$

Proposition 2.3.5 *Let $A \in M_n$. The following are equivalent.*

(a) *A is positive semidefinite.*

(b) *A is unitarily similar to a real diagonal matrix with nonnegative diagonal entries.*

(c) *$A = B^*B$ for some $B \in M_n$. (We can choose B so that $B = B^*$.)*

Proof. Suppose (a) holds. Then $\mathbf{x}^*A\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{C}^n$. Thus, there is a unitary $U \in M_n$ such that $U^*AU = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)$ with $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. If there is $\lambda_j < 0$, we can let $\mathbf{x}$ be the jth column of U so that $\mathbf{x}^*A\mathbf{x} = \lambda_j < 0$, which is a contradiction. So, all $\lambda_1, \ldots, \lambda_n \geq 0$.

Suppose (b) holds. Then $U^*AU = D$ such that D has nonnegative entries. We have $A = B^*B$ with $B = UD^{1/2}U^* = B^*$. Hence condition (c) holds.

Suppose (c) holds. Then for any $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{x}^*A\mathbf{x} = (B\mathbf{x})^*(B\mathbf{x}) \geq 0$. Thus, (a) holds. □

A quick proof of SVD and an efficient algorithm to find SVD.

Let $A \in M_{m,n}$. Then A^*A is psd so that $V^*A^*AV = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)$. Since $\lambda_j = v_jA^*Av_j$, we see that $\lambda_j = s_j^2$ for some $s_j \geq 0$, and we may assume that $s_1^2 \geq \cdots \geq s_n^2$. Let $s_1^2, \ldots, s_r^2$ be the nonzero eigenvalues of A^*A, and let $u_j = A_jv_j/\|A_jv_j\| \in \mathbb{C}^m$ for $j = 1, \ldots, r$. Then $\{u_1, \ldots, u_r\}$ is an orthonormal set and $A = \sum_{j=1} s_ju_jv_j^*$. If we only need $A_k = \sum_{j=1}^k s_ju_jv_j^*$, one can use power method to get s_1, v_1 and then u_1 from A^*A. Then get s_2, v_2 and then u_2 from $A_2^*A_2$ with $A_2 = A - s_1u_1v_1^*$, and so forth.
For any $A \in M_n$ we can write $A = H + iG$ with $H = (A + A^*)/2$ and $G = (A - A^*)/(2i)$. This is known as the Hermitian or Cartesian decomposition.

Theorem 2.3.6 *Let $A \in M_n$. Then $A = PU = VQ$ for some positive semidefinite matrices $P, Q \in M_n$ and unitary $U, V \in M_n$.*

- *If A is invertible, then the matrices P, Q, U, V are uniquely determined as $(P, U) = (\sqrt{AA^*}, P^{-1}A)$, and $(Q, V) = \sqrt{A^*A}, AQ^{-1})$.*
- *The matrix A is normal if and only if $PU = UP$ or $VQ = QV$.*

Corollary 2.3.7 *In fact, if $A \in M_{n,m}$ with $n \geq m$ and has rank m, then $A = VR$ where $V \in M_{n,m}$ has orthonormal columns and $R \in M_m$ can be chosen to be upper triangular, lower triangular, or positive definite.*

2.4 Commuting families and Specht's theorem

Definition 2.4.1 *A family $\mathcal{F} \subseteq M_n$ is a commuting family if every pair of matrices $X, Y \in \mathcal{F}$ commute, i.e., $XY = YX$.*

Lemma 2.4.2 *Let $\mathcal{F} \subseteq M_n$ be a commuting family. Then there a unit vector $v \in \mathbb{C}^n$ such that v is an eigenvector for every $A \in \mathcal{F}$.*

Proof. Let $V \subseteq \mathbb{C}^n$ with minimum positive dimension be such that $A(V) \subseteq V$. We will show that $\dim V = 1$ and the result will follow. First, $A(\mathbb{C}^n) \subseteq \mathbb{C}^n$. So, one can always try to find V with a minimum positive dimension. We claim that every nonzero vector in V is an eigenvector of A for every $A \in \mathcal{F}$. Then for any non-zero $v \in V$, $V_0 = \text{span}\,\{v\}$ will satisfy $A(V_0) \subseteq V_0$ with $\dim V_0 = 1$.

Suppose there is $A \in \mathcal{F}$ such that not every nonzero vector in v is an eigenvector of A. Now, if V has an orthonormal basis $\{u_1, \ldots, u_k\}$ and U is unitary with $u_1, \ldots, u_k$ as the first k columns. Then $U^*BU = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix}$ with $B \in M_k$ for every $B \in \mathcal{F}$. Then there is $v = a_1u_1 + \cdots + a_ku_k \in V$ such that $Av = \lambda v$.

Let $V_0 = \{u \in V : Au = \lambda u\} \subset V$. Then V_0 is a subspace of V with smaller dimension. Next, we show that $Bu \in V_0$ for any $u \in V_0$. If $B \in \mathcal{F}$ and $u \in V$, then $Bu \in V$ as $B(V) \subseteq V$, and $A(Bu) = BAu = B\lambda u = \lambda Bu$, i.e., $\tilde{u} = Bu \in V_0$. So, V_0 satisfies $B(V_0) \subseteq V_0$ and $\dim V_0 < \dim V$, which is impossible. The desired result follows. □

Theorem 2.4.3 *Let $\mathcal{F} \subseteq M_n$ be a commuting family. Then there is a unitary matrix $U \in M_n$ such that U^*AU is in upper triangular form.*

Proof. We can consider the a basis for the span of $\mathcal{F}$, and assume that $\mathcal{F} = \{A_1, \ldots, A_m\}$ is finite. Assume A_1 is nonscalar, and has an eigenvalue λ_1. Then $A_j(\mathbf{V}) \subset \mathbf{V}$ if $\mathbf{V}$ is the null space of $A_1 - \lambda_1 I$. By induction, there is a common unit eigenvector x for all $A_j \in \mathcal{F}$. Then construct U with x as the first column so that $U^*A_jU = \begin{pmatrix} * & * \\ 0 & B_j \end{pmatrix}$, where $\{B_1, \ldots, B_m\}$ is a commuting families. Apply induction to finish the proof. □

Corollary 2.4.4 *Suppose $\mathcal{F} \subseteq M_n$ is a commuting family of normal matrices. Then there is a unitary matrix $U \in M_n$ such that U^*AU is in diagonal form.*

There is no easy canonical form under unitary similarity.[1] How to determine two matrices are unitarily similar?

Definition 2.4.5 *Let $\{X, Y\} \subseteq M_n$. A word $W(X, Y)$ in X and Y of length m is a product of m matrices chosen from $\{X, Y\}$ (with repetition).*

Theorem 2.4.6 *Let $A, B \in M_n$.*

(a) *If A and B are unitarily similar, then $\text{tr}\,(W(A, A^*)) = \text{tr}\,(W(B, B^*))$ for all words $W(X, Y)$.*

(b) $\text{tr}\,(W(A, A^*)) = \text{tr}\,(W(B, B^*))$ *for all words $W(X, Y)$ of length $2n^2$, then A and B are unitarily similar.*

[1]Helene Shapiro, A survey of canonical forms and invariants for unitary similarity, Linear Algebra Appl. 147 (1991), 101-167.

2.5 Other canonical forms

Unitary congruence

- A matrix $A \in M_n$ is unitarily congruent to $B \in M_n$ if there is a unitary matrix U such that $A = U^t B U$.
- There is no easy canonical form under unitary congruence for general matrices.
- Every complex symmetric matrix $A \in M_n$ is unitarily congruent to $\sum_{j=1}^k s_j E_{jj}$, where $s_1 \geq \cdots \geq s_k > 0$ are the nonzero singular values of A.
- Every skew-symmetric $A \in M_n$ is unitarily congruent to 0_{n-2k} and

$$\begin{pmatrix} 0 & s_j \\ -s_j & 0 \end{pmatrix}, \qquad j = 1, \ldots, k,$$

  where $s_1 \geq \cdots \geq s_k > 0$ are nonzero singular values of A.

- The singular values of a skew-symmetric matrix $A \in M_n$ occur in pairs.
- Two symmetric (skew-symmetric) matrices are unitarily congruent if and only if they have the same singular values.

Proof. Suppose $A \in M_n$ is symmetric. Let $\mathbf{x} \in \mathbb{C}^n$ be a unit vector so that $\mathbf{x}^t A \mathbf{x}$ is real and maximum, and let $U \in M_n$ be unitary with $\mathbf{x}$ as the first column. Show that $U^t A U = [s_1] \oplus A_1$. Then use induction.

Suppose $A \in M_n$ is skew-symmetric. Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ be orthonormal pairs such that $\mathbf{x}^t A \mathbf{y}$ is real and maximum, and $U \in M_n$ be unitary with $\mathbf{x}, \mathbf{y}$ as the first two columns. Show that $U^t A U = \begin{pmatrix} 0 & s_1 \\ -s_1 & 0 \end{pmatrix} \oplus A_1$. Then use induction. □

2.6 Real matrices

Theorem 2.6.1 *Let $A \in M_n$ be a real matrix, and*

$$\det(xI - A) = (x - c_1) \cdots (x - c_r)(x^2 - 2a_1 x + a_1^2 + b_1^2) \cdots (x^2 - 2a_k x + a_k^2 + b_k^2).$$

Then there is an real orthogonal matrix P such that $P^t A P = (C_{rs})_{0 \leq r,s \leq k}$ is in upper triangular block form, where $C_{00} \in M_r(\mathbb{R})$ is an upper triangular matrix with diagonal entries $c_1, \ldots, c_r$, $C_{jj} \in M_2(\mathbb{R})$ has eigenvalues $a_j \pm i b_j$ for $j = 1, \ldots, k$, and C_{rs} is zero if $r > s$.

Furthermore, if A is normal, i.e., $A^tA = AA^t$, then

$$P^tAP = B_0 \oplus B_1 \oplus \cdots \oplus B_k$$

with $B_0 = \text{diag}\,(c_1, \ldots, c_r)$, *and* $B_j = \begin{pmatrix} a_j & b_j \\ -b_j & a_j \end{pmatrix} \in M_2(\mathbb{R})$ *for* $j = 1, \ldots, k$.

(a) *If $A = A^t$, then $B_1, \ldots, B_k$ are vacuous.*
(b) *if $A = -A^t$, then $B_0 = 0_r$.*
(c) *If A is orthogonal, then $b_1, \ldots, b_r \in \{1, -1\}$ and $a_j^2 + b_j^2 = 1$ for $j = 1, \ldots, k$.*

Proof. If A has a real eigenvalue c_1 and $Au_1 = c_1u_1$, where u_1 is a unit eigenvector. Let P be real orthogonal with u_1 as the first column. Then $P_1^tAP_1 = \begin{pmatrix} c_1 & \star \\ 0 & A_1 \end{pmatrix}$. If A has another real eigenvalue c_2, then A_1 has c_1 as an eigenvalue and there is an orthogonal matrix $P_2 \in M_{n-1}$ such that $P_2^tA_1P_2 = \begin{pmatrix} c_2 & \star \\ 0 & A_2 \end{pmatrix}$. Then

$$([1] \oplus P_2^t)P_1^tAP_1([1] \oplus P_2) = \begin{pmatrix} c_1 & \star & \star \\ 0 & c_2 & \star \\ 0 & 0 & A_2 \end{pmatrix}.$$

Repeating this argument, we can get

$$P_r^tAP_r = \begin{pmatrix} C_{00} & \star \\ 0 & C_1 \end{pmatrix}.$$

Now, C_1 has complex eigenvalue $a_1 \pm ib_1$. If $C_1(x + iy) = (a_1 + ib_1)(x + iy)$ for a pair of nonzero real vectors $x, y \in \mathbb{R}^n$. Then $C_1x = a_1x - b_1y$ and $C_1y = a_1y + b_1x$, and $C_1(x - iy) = (a_1 - ib_1)(x - iy)$, i.e., $C_1[x\ y] = [x\ y]B_1$. Now, $x + iy$ and $x - iy$ are eigenvectors of B_1 corresponding to the eigenvalues $a_1 \pm ib_1$. So, $\{x + iy, x - iy\}$ is linear independent and so is $\{x, y\}$. Apply Gram-Schmidt process to $\{x, y\}$ to get a real orthonormal family $\{q_1, q_2\}$. Then $[x\ y] = [q_1\ q_2]T_1$ for an upper triangular matrix $T_1 \in M_2(\mathbb{R})$. Let $Q_1 \in M_{2k}$ be real orthogonal with q_1, q_2 as the first two columns. Then

$$Q_1^tB_1Q_1 = \begin{pmatrix} C_{11} & \star \\ 0 & C_2 \end{pmatrix}$$

so that $C_{11} = T_1B_1T_1^{-1}$ has eigenvalues $a_1 \pm ib_1$. One can apply an inductive arguments to C_2 and get the desired form.

In case A is normal, then so is Q^tAQ. One can then deduce that Q^tAQ has the form $B_0 \oplus \cdots \oplus B_k$. Assertions (a) – (c) can be verified directly. □

3 Similarity and equivalence

We consider other canonical forms in this chapter.

3.1 Jordan Canonical form

Theorem 3.1.1 *Suppose $A \in M_n$ has distinct eigenvalues $\lambda_1, \dots, \lambda_k$. Then A is similar to $A_{11} \oplus \cdots \oplus A_{kk}$ such that A_{jj} has (only one distinct) eigenvalue λ_j for $j = 1, \dots, k$.*

Lemma 3.1.2 *Suppose $A \in M_m, B \in M_n$ have no common eigenvalues. Then for any $C \in M_{m,n}$ there is a unique solution $X \in M_{m,n}$ such that $AX - XB = C$.*

Proof. Let U be unitary such that $\tilde{B} = U^*BU$ is in upper triangular form. If $\tilde{C} = CU$ and $Y = XU$, then we consider $\tilde{C} = CU = A(XU) - (XU)U^*BU = AY - Y\tilde{B}$ and solve for Y. Let $\tilde{C} = [c_1 \cdots c_n], Y = [y_1 \cdots y_n]$ and $\tilde{B} = [b_{ij}]$, where $b_{11}, \dots, b_{nn}$ are the eigenvalues of B. Then

$c_1 = Ay_1 - b_{11}y_1$ has a unique solution y_1 as $A - b_{11}I$ is invertible,

$c_2 = Ay_2 - b_{22}y_2 - b_{12}y_1$ has a unique a solution y_2 as $A - b_{22}I$ is invertible,

....

$c_n = Ay_n - b_{nn}y_n - \sum_{j=1}^{n-1} b_{1j}y_j$ has a unique solution y_n as $A - b_{nn}I$ is invertible. □

Proposition 3.1.3 *Suppose $A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \in M_n$ such that $A_{11} \in M_k, A_{22} \in M_{n-k}$ have no common eigenvalue. Then A is similar to $A_{11} \oplus A_{22}$.*

Proof. By the previous lemma, there is X be such that $A_{11}X + A_{12} = XA_{22}$. Let $S = \begin{pmatrix} I_k & X \\ 0 & I_{n-k} \end{pmatrix}$ so that $AS = S(A_{11} \oplus A_{22})$. The result follows. □

Definition 3.1.4 *Let $J_k(\lambda) \in M_k$ such that all the diagonal entries equal λ and all super diagonal entries equal 1. Then $J_k(\lambda) = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \\ & & & \lambda \end{pmatrix} \in M_k$ is call a (an upper triangular) Jordan block of λ of size k.*

Theorem 3.1.5 *Every $A \in M_n$ is similar to a direct sum of Jordan blocks.*

Proof. We may assume that $A = A_{11} \oplus \cdots \oplus A_{kk}$. If we can find invertible matrices $S_1, \ldots, S_k$ such that $S_i^{-1} A_{ii} S_i$ is in Jordan form, then $S^{-1}AS$ is in Jordan form for $S = S_1 \oplus \cdots \oplus S_k$.

Focus on $T = A_{ii} - \lambda_i I_{n_k}$. If $S^{-1}TS$ is in Jordan form, then so is A_{ii}.

One may see http://cklixx.people.wm.edu/teaching/math408/Jordan.pdf for a proof of this. The note will appear on arXiv soon. □

To determine the Jordan form of a matrix A with $\det(xI - A) = (x-\lambda_1)^{n_1} \cdots (x - \lambda_k)^{n_k}$, one only needs to study the rank of $(A - \lambda_j I)^m$ for $m = 1, \ldots, n_j$.

Let $\ker((A-\lambda I)^i) = \ell_i$ has dimension ℓ_i. Then there are ℓ_1 Jordan blocks of λ, and there are $\ell_i - \lambda_{i-1}$ Jordan blocks of size at least i.

Example 3.1.6 *Let* $T = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. *Then* $Te_1 = 0, Te_2 = 0, Te_3 = e_1 + 3e_2, Te_4 = 2e_1 + 4e_2$. *So,* $T(V) = \text{span}\,\{e_1, e_2\}$. *Now,* $Te_1 = Te_2 = 0$ *so that* e_1, e_2 *form a Jordan basis for* $T(V)$. *Solving* u_1, u_2 *such that* $T(u_1) = e_1, T(u_2) = e_2$, *we let* $u_1 = -2e_3 + 3e_4/2$ *and* $u_2 = e_3 - e_4/2$. *Thus,* $TS = S(J_2(0) \oplus J_2(0))$ *with*

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \\ 0 & 3/2 & 0 & -1/2 \end{pmatrix}.$$

Example 3.1.7 *Let* $T = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$. *Then* $Te_1 = 0, Te_2 = e_1, Te_3 = 2e_1 + e_2$. *So,* $T(V) = \text{span}\,\{e_1, e_2\}$, *and* $e_2, Te_2 = e_1$ *form a Jordan basis for* $T(V)$. *Solving* u_1 *such that* $T(u_1) = e_2$, *we have* $u_1 = (-2e_2 + e_3)/3$. *Thus,* $TS = SJ_3(0)$ *with*

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1/3 \end{pmatrix}.$$

Example 3.1.8 *Suppose* $A \in M_9$ *has distinct eigenvalues* $\lambda_1, \lambda_2, \lambda_3$ *such that* $A - \lambda_1 I$ *has rank 8,* $A - \lambda_2 I$ *has rank 7,* $(A - \lambda_2 I)^2$ *and* $(A - \lambda_2 I)^3$ *have rank 5,* $A - \lambda_3 I$ *has rank 6,* $(A - \lambda_3 I)^2$ *and* $(A - \lambda_3 I)^3$ *have rank 5. Then the Jordan form of* A *is*

$$J_1(\lambda_1) \oplus J_2(\lambda_2) \oplus J_2(\lambda_2) \oplus J_1(\lambda_3) \oplus J_1(\lambda_3) \oplus J_2(\lambda_3).$$

3.2 Implications of the Jordan form

Theorem 3.2.1 *Two matrices are similar if and only if they have the same Jordan form.*

Proof. If A and B have Jordan form J, then $S^{-1}AS = J = T^{-1}BT$ for some invertible S, T so that $R^{-1}AR = B$ with $R = ST^{-1}$.

If $S^{-1}AS = B$, then $\operatorname{rank}(A-\mu I)^\ell = \operatorname{rank}(B-\mu I)^\ell$ for all eigenvalues of A or B, and for all positive integers ℓ. So, A and B have the same Jordan form. $\square$

Remark 3.2.2 *If $A = S(J_1 \oplus \cdots \oplus J_k)S^{-1}$, then $A^m = S(J_1^m \oplus \cdots \oplus J_k^m)S^{-1}$.*

Theorem 3.2.3 *Let $J_k(\lambda) = \lambda I_k + N_k$, where $N_k = \sum_{j=1}^{k-1} E_{j,j+1}$. Then*

$$J_k(\lambda)^m = \sum_{j=0}^{m} \binom{m}{j} \lambda^{m-j} N_k^j,$$

where $N_k^0 = I_k$, $N_k^j = 0$ for $j \ge k$, and N_k^j has one's at the jth super diagonal (entries with indexes $(\ell, \ell+j)$) and zeros elsewhere.

For every polynomial function $f(z) = a_m z^m + \cdots + a_0$, let

$$f(A) = a_m A^m + \cdots + a_0 I_n \quad \text{for } A \in M_n.$$

Definition 3.2.4 *Let $A \in M_n$. Then there is a unique monic polynomial*

$$m_A(z) = x^m + a_1 x^{m-1} + \cdots + a_m$$

such that $m_A(A) = 0$. It is called the minimal polynomial of A.

Theorem 3.2.5 *A polynomial $g(z)$ satisfies $g(A) = 0$ if and only if it is a multiple of the minimal polynomial of A.*

Proof. If $g(z) = m_A(z)q(z)$, then $g(A) = m_A(A)q(A) = 0$. To prove the converse, by the Euclidean algorithm, $g(z) = m_A(z)q(z) + r(z)$ for any polynomial $g(z)$. If $0 = g(A) = m_A(A)q(A) + r(A) = r(A)$, then $r(A) = 0$. But $r(z)$ has degree less than $m_A(z)$. If $r(z)$ is not zero, then there is a nonzero $\mu \in \mathbb{C}$ such that $\mu r(z)$ is a monic polynomial such that $\mu r(A) = 0$, which is impossible. So, $r(z) = 0$, i.e., $g(z)$ is a multiple of $m_A(z)$. $\square$

We can actually determine the minimal polynomial of $A \in M_n$ using its Jordan form.

Theorem 3.2.6 *Suppose A has distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ such that r_j is the maximum size Jordan block of λ_j for $j = 1, \ldots, k$. Then $m_A(z) = (z-\lambda_1)^{r_1} \cdots (z-\lambda_k)^{m_k}$.*

Proof. Following the proof of the Cayley Hamilton Theorem, we see that $m_A(A) = 0_n$. By the last Theorem, if $g(A) = 0_n$, then $g(z) = m_A(z)q(z)$. So, taking $q(z) = 1$ will yield the monic polynomial of minimum degree satisfying $m_A(A) = 0$. □

Remark 3.2.7 *For any polynomial $g(z)$, the Jordan form of $g(A)$ can be determine in terms of the Jordan form of A. In particular, for every Jordan block $J_k(\lambda)$, we can write $g(z) = (z-\lambda)^k q(z) + r(z)$ with $r(z) = a_0 + \cdots + a_{k-1}z^{k-1}$ so that $g(J_k(\lambda)) = r(J_k(\lambda))$.*

Note that

$$g(J_r(\lambda)) = \begin{pmatrix} \frac{g(\lambda)}{0!} & \frac{g'(\lambda)}{1!} & \frac{g''(\lambda)}{2!} & \cdots & \frac{g^{(r-1)}(\lambda)}{(r-1)!} \\ 0 & \frac{g(\lambda)}{0!} & \frac{g'(\lambda)}{1!} & \vdots & \frac{g^{(r-2)}(\lambda)}{(r-2)!} \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \cdots & \ddots & \frac{g(\lambda)}{0!} & \frac{g'(\lambda)}{1!} \\ 0 & \cdots & \cdots & 0 & \frac{g(\lambda)}{0!} \end{pmatrix}.$$

One can extend this to function $g(x)$, which are differentiable up to order r in a domain containing λ in the interior.

3.3 Further canonical forms

Equivalence

- Two matrices $A, B \in M_{m,n}$ are equivalent if there are invertible matrices $R \in M_m, S \in M_n$ such that $A = RBS$.
- Every matrix $A \in M_{m,n}$ is equivalent to $\sum_{j=1}^{k} E_{jj}$, where k is the rank of A.
- Two matrices are equivalent if they have the same rank.

Proof. Elementary row operations and elementary column operations. □

$*$-congruence

- A matrix $A \in M_n$ is $*$-congruent to $B \in M_n$ if there is an invertible matrix S such that $A = S^*BS$.
- There is no easy canonical form under $*$-congruence for general matrix.[2]
- Every Hermitian matrix $A \in M_n$ is $*$-congruent to $I_p \oplus -I_q \oplus 0_{n-p-q}$. The triple $\nu(A) = (p, q, n-p-q)$ is known as the inertia of A.

[2]Roger A. Horn and Vladimir V. Sergeichuk, Canonical forms for complex matrix congruence and $*$-congruence, Linear Algebra Appl. (2006), 1010-1032.

- Two Hermitian matrices are $*$-congruent if and only if they have the same inertia.

Proof. Use the unitary congruence/similarity results. $\square$

Congruence or t-congruence

- A matrix $A \in M_n$ is t-congruent to $B \in M_n$ if there is an invertible matrix S such that $A = S^t BS$.
- There is no easy canonical form under t-congruence for general matrices; see footnote 2.
- Every complex symmetric matrix $A \in M_n$ is t-congruent to $I_k \oplus 0_{n-k}$, where $k = \text{rank}\,(A)$.
- Every skew-symmetric $A \in M_n$ is t-congruent to 0_{n-2k} and k copies of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
- The rank of a skew-symmetric matrix $A \in M_n$ is even.
- Two symmetric (skew-symmetric) matrices are t-congruent if and only if they have the same rank.

Proof. Use the unitary congruence results. $\square$

3.4 Remarks on real matrices

Remark 3.4.1 *Let $A \in M_n(\mathbb{R})$. Then $A = S + K$ where $S = (A + A^t)/2$ is symmetric and $K = (A - A^t)/2$ is skew-symmetric, i.e., $K^t = -K$.*

- *Note that $x^t Kx = 0$ for all $x \in \mathbb{R}^n$.*
- *Clearly, $x^t Ax \in \mathbb{R}$ for all real vectors $x \in \mathbb{R}^n$, and the condition does not imply that A is symmetric as in the complex Hermitian case.*
- *The matrix A satisfies $x^t Ax \geq 0$ for all if and only if $(A + A^t)/2$ has only nonnegative eigenvalues. The condition does not automatically imply that A is symmetric as in the complex Hermitian case.*
- *Every skew-symmetric matrix $K \in M_n(\mathbb{R})$ is orthogonally similar to 0_{2k} and*

$$\begin{pmatrix} 0 & s_j \\ -s_j & 0 \end{pmatrix}, \qquad j = 1, \ldots, k,$$

  where $s_1 \geq \cdots \geq s_k > 0$ are nonzero singular values of A.

- If $A \in M_n(\mathbb{R})$ has only real eigenvalues, then one can find a real invertible matrix such that $S^{-1}AS$ is in Jordan form.

- If $A \in M_n(\mathbb{R})$, then there is a real invertible matrix such that $S^{-1}AS$ is a direct sum of real Jordan blocks, and $2k \times 2k$ generalized Jordan blocks of the form $(C_{ij})_{1\le i,j\le k}$ with $C_{11} = \cdots = C_{kk} = \begin{pmatrix} \mu_1 & \mu_2 \\ -\mu_2 & \mu_1 \end{pmatrix}$, $C_{12} = \cdots = C_{k-1,k} = I_2$, and all other blocks equal to 0_2.

- The proof can be done by the following two steps.

  First of all, find the Jordan form of A. Then group $J_k(\lambda)$ and $J_k(\bar{\lambda})$ together for any complex eigenvalues, and find a complex S such that $S^{-1}AS$ is a direct sum of the above form.

  Second if $S = S_1 + iS_2$ for some real matrix S_1, S_2, show that there is $\hat{S} = S_1 + rS_2$ for some real number r such that $\hat{S}$ is invertible so that $\hat{S}^{-1}A\hat{S}$ has the desired form.

4 Eigenvalues and singular values inequalities

We study inequalities relating the eigenvalues, diagonal elements, singular values of matrices in this chapter.

For a Hermitian matrix A, let $\lambda(A) = (\lambda_1(A), \dots, \lambda_n(A))$ be the vector of eigenvalues of A with entries arranged in descending order. Also, we will denote by $s(A) = (s_1(A), \dots, s_n(A))$ the singular values of a matrix $A \in M_{m,n}$. For two Hermitian matrices, we write $A \geq B$ if $A - B$ is positive semidefinite.

4.1 Diagonal entries and eigenvalues of a Hermitian matrix

Theorem *Let $A = (a_{ij}) \in M_n$ be Hermitian with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Then for any $1 \leq k < n$, $a_{11} + \cdots + a_{kk} \leq \lambda_1 + \cdots + \lambda_k$. The equality holds if and only if $A = A_{11} \oplus A_{22}$ so that A_{11} has eigenvalues $\lambda_1, \dots, \lambda_k$.*

Remark The above result will give us what we needed, and we can put the majorization result as a related result for real vectors.

Lemma 4.1.1 (Rayleigh principle) *Let $A \in M_n$ be Hermitian. Then for any unit vector $\mathbf{x} \in \mathbb{C}^n$,*

$$\lambda_1(A) \geq \mathbf{x}^* A \mathbf{x} \geq \lambda_n(A).$$

The equalities hold at unit eigenvectors corresponding to the largest and smallest eigenvalues of A, respectively.

Proof. Done in homework problem. □

If we take $\mathbf{x} = e_j$, we see that every diagonal entry of a Hermitian matrix A lies between $\lambda_1(A)$ and $\lambda_n(A)$.

We can say more in the following. To do that we need the notion of **majorization** and **doubly stochastic matrices**.

A matrix $D = (d_{ij}) \in M_n$ is doubly stochastic if $d_{ij} \geq 0$ and all the row sums and column sums of D equal 1.

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. We say that $\mathbf{x}$ is weakly majorized by $\mathbf{y}$, denoted by $\mathbf{x} \prec_w \mathbf{y}$ if the sum of the k largest entries of $\mathbf{x}$ is not larger than that of $\mathbf{y}$ for $k = 1, \dots, n$; in addition, if the sum of the entries of $\mathbf{x}$ and $\mathbf{y}$, we say that $\mathbf{x}$ is majorized by $\mathbf{y}$, denoted by $\mathbf{x} \prec \mathbf{y}$. We say that $\mathbf{x}$ is obtained from $\mathbf{y}$ be a pinching if $\mathbf{x}$ is obtained from $\mathbf{y}$ by changing (y_i, y_j) to $(y_i - \delta, y_j + \delta)$ for two of the entries $y_i > j_j$ of y and some $\delta \in (0, y_i - y_j)$.

Theorem 4.1.2 *Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ with $n \geq 2$. The following conditions are equivalent.*

(a) $\mathbf{x} \prec \mathbf{y}$.

(b) *There are vectors* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ *with* $k < n$, $\mathbf{x}_1 = \mathbf{y}$, $\mathbf{x}_k = \mathbf{x}$, *such that each* $\mathbf{x}_j$ *is obtained from* $\mathbf{x}_{j-1}$ *by pinching two of its entries.*

(c) $\mathbf{x} = D\mathbf{y}$ *for some doubly stochastic matrix.*

Proof. Note that the conditions do not change if we replace $(\mathbf{x}, \mathbf{y})$ by $(P\mathbf{x}, Q\mathbf{y})$ for any permutation matrices P, Q. We may make these changes in our proof.

(c) $\Rightarrow$ (a). We may assume that $\mathbf{x} = (x_1, \ldots, x_n)^t$ and $\mathbf{y} = (y_1, \ldots, y_n)^t$ with entries in descending order. Suppose $\mathbf{x} = D\mathbf{y}$ for a doubly stochastic matrix $D = (d_{ij})$. Let $\mathbf{v}_k = (e_1 + \cdots + e_k)$ and $\mathbf{v}_k^t D = (c_1, \ldots, c_n)$. Then $0 \le c_j \le 1$ and $\sum_{j=1}^n c_j = k$. So,

$$\begin{aligned}\sum_{j=1}^k x_j &= \mathbf{v}_k^t D y = c_1 y_1 + \cdots + c_n y_n \\ &\le c_1 y_1 + c_k y_k + [(1 - c_1) + \cdots + [1 - c_k)] y_k \le y_1 + \cdots + y_k.\end{aligned}$$

Clearly, the equality holds if $k = n$.

(a) $\Rightarrow$ (b). We prove the result by induction on n. If $n = 2$, the result is clear. Suppose the result holds for vectors of length less than n. Assume $\mathbf{x} = (x_1, \ldots, x_n)^t$ and $\mathbf{y} = (y_1, \ldots, y_n)^t$ has entries arranged in descending order, and $\mathbf{x} \prec \mathbf{y}$. Let k be the maximum integer such that $y_k \ge x_1$. If $k = n$, then for $S = \sum_{j=1}^n x_j = \sum_{j=1}^n y_j$,

$$y_n \ge x_1 \ge \cdots x_n \ge= S - \sum_{j=1}^{n-1} x_j \ge S - \sum_{j=1}^{n-1} y_j = y_n$$

so that $x = \cdots = x_n = y_1 = \cdots = y_n$. So, $\mathbf{x} = \mathbf{x}_1 = \mathbf{y}$. Suppose $k < n$ and $y_k \ge x_1 > y_{k+1}$. Then we can replace (y_k, y_{k+1}) by $(\tilde{y}_k, \tilde{y}_{k+1}) = (x_1, y_k + y_{k+1} - x_1)$. Then removing x_1 from $\mathbf{x}$ and removing $\tilde{y}_k$ in $\mathbf{x}_1$ will yield the vectors $\tilde{\mathbf{x}} = (x_2, \ldots, x_n)^t$ and $\tilde{\mathbf{y}} = (y_1, \ldots, y_{k-1}, \tilde{y}_{k+1}, \ldots, y_n)^t$ in $\mathbb{R}^{n-1}$ with entries arranged in descending order. We will show that $\tilde{\mathbf{x}} \prec \tilde{\mathbf{y}}$. The result will then follows by induction. Now, if $\ell \le k$, then

$$x_2 + \cdots + x_\ell \le x_1 + \cdots + x_{\ell-1} \le y_1 + \cdots + y_{\ell-1};$$

if $\ell > k$, then

$$x_2 + \cdots + x_\ell \le (y_1 + \cdots + y_\ell) - x_1 = y_1 + \cdots + y_{k-1} + \tilde{y}_{k+1} + y_{k+1} + \cdots + y_\ell$$

with equality when $\ell = n$. The results follows.

(b) $\Rightarrow$ (c). If $\mathbf{x}_j$ is obtained from $\mathbf{x}_{j-1}$ by pinching the pth and qth entries. Then there is a doubly stochastic matrix P_j obtained from I by changing the submatrix in rows and columns p, q by

$$\begin{pmatrix} t_j & 1-t_j \\ 1-t_j & t_j \end{pmatrix}$$

for some $t_j \in (0,1)$. Then $\mathbf{x} = D\mathbf{y}$ for $D = P_k \cdots P_1$, which is doubly stochastic. □

Theorem 4.1.3 *Let $\mathbf{d}, \mathbf{a} \in \mathbb{R}^n$. The following are equivalent.*

(a) *There is a complex Hermitian (real symmetric) $A \in M_n$ with entries of $\mathbf{a}$ as eigenvalues and entries of $\mathbf{d}$ as diagonal entries.*

(b) *The vectors satisfy $\mathbf{d} \prec \mathbf{a}$.*

Proof. Let $A = UDU^*$ such that $D = \mathrm{diag}\,(\lambda_1, \dots, \lambda_n))$. Suppose $A = (a_{ij})$ and $U = (u_{ij})$. Then $a_{jj} = \sum_{i=1}^n \lambda_i |u_{ji}|^2$. Because $(|u_{ji}|^2)$ is doubly stochastic. So, $(a_{11}, \dots, a_{nn}) \prec (\lambda_1, \dots, \lambda_n)$.

We prove the converse by induction on n. Suppose $(d_1, \dots, d_n) \prec (\lambda_1, \dots, \lambda_n)$. If $n = 2$, let $d_1 = \lambda_1 \cos^2\theta + \lambda_2 \sin^2\theta$ so that

$$(a_{ij}) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

has diagonal entries d_1, d_2.

Suppose $n > 2$. Choose the maximum k such that $\lambda_k \geq d_1$. If $\lambda_n = d_1$, then for $S = \sum_{j=1}^n d_j = \sum_{j=1}^n \lambda_j$ we have

$$\lambda_n \geq d_1 \geq \cdots \geq d_n = S - \sum_{j=1}^{n-1} d_j \geq S - \sum_{j=1}^{n-1} \lambda_j = \lambda_n.$$

Thus, $\lambda_n = d_1 = \cdots = d_n = S/n = \sum_{j=1}^n \lambda_j/n$ implies that $\lambda_1 = \cdots = \lambda_n$. Hence, $A = \lambda_n I$ is the required matrix. Suppose $k < n$. Then there is $A_1 = A_1^t \in M_2(\mathbb{R})$ with diagonal entries $d_1, \lambda_k + \lambda_{k+1} - d_1$ and eigenvalues λ_j, λ_{j+1}. Consider $A = A_1 \oplus D$ with $D = \mathrm{diag}\,(\lambda_1, \dots, \lambda_{k-1}, \lambda_{k+2}, \dots, \lambda_n)$. As shown in the proof of Theorem 4.1.3, if $\tilde{\lambda}_{k+1} = \lambda_k + \lambda_{k+1} - d_1$, then

$$(d_2, \dots, d_n) \prec (\tilde{\lambda}_{k+1}, \lambda_1, \dots, \lambda_{k-1}, \lambda_{k+2}, \dots, \lambda_n).$$

By induction assumption, there is a unitary $U \in M_{n-1}$ such that

$$U([\tilde{\lambda}_k] \oplus D)U^* \in M_{n-1}$$

has diagonal entries $d_2, \dots, d_n$. Thus, $A = ([1] \oplus U)(A_1 \oplus D)([1] \oplus U^*)$ has the desired eigenvalues and diagonal entries. □

4.2 Max-Min and Min-Max characterization of eigenvalues

In this subsection, we give a Max-Min and Min-Max characterization of eigenvalues of a Hermitian matrix.

Lemma 4.2.1 *Let V_1 and V_2 be subspaces of $\mathbb{C}^n$ such that $\dim(V_1) + \dim(V_2) > n$, then $V_1 \cap V_2 \neq \{0\}$.*

Proof. Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ and $\{\mathbf{v}_1, \ldots, \mathbf{v}_q\}$ be bases for V_1 and V_2. Then $p+q > n$ and the linear system $[\mathbf{u}_1 \cdots \mathbf{u}_p \mathbf{v}_1 \cdots \mathbf{v}_q]\mathbf{x} = \mathbf{0} \in \mathbb{C}^n$ has a non-trivial solution $\mathbf{x} = (x_1, \ldots, x_p, y_1, \ldots, y_q)^t$. Note that not all $x_1, \ldots, x_p$ are zero, else $y_1\mathbf{v}_1 + \cdots + y_q\mathbf{v}_1 = 0$ implies $y_j = 0$ for all j. Thus, $\mathbf{v} = x_1\mathbf{u}_1 + \cdots + x_p\mathbf{u}_p = -(y_1\mathbf{v}_1 + \cdots + y_q\mathbf{v}_q)$ is a nonzero vector in $\mathbf{V}_1 \cap \mathbf{V}_2$. □

Theorem 4.2.2 *Let $A \in M_n$ be Hermitian. Then for $1 \leq k \leq n$,*

$$\begin{aligned}\lambda_k(A) &= \max\{\lambda_k(X^*AX) : X \in M_{n,k}, X^*X = I_k\} \\ &= \min\{\lambda_1(Y^*AY) : Y \in M_{n,n-k+1}, Y^*Y = I_{n-k+1}\}.\end{aligned}$$

Equivalently,

$$\lambda_k(A) = \max_{\substack{V \leq \mathbb{C}^n \\ \dim V = k}} \min_{\substack{x \in V \\ \|x\| = 1}} x^*Ax = \min_{\substack{V \leq \mathbb{C}^n \\ \dim V = n-k+1}} \max_{\substack{x \in V \\ \|x\| = 1}} x^*Ax.$$

Proof. Suppose $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is a family of orthonormal eigenvectors of A corresponding to the eigenvalues $\lambda_1(A), \ldots, \lambda_n(A)$. Let $X = [\mathbf{u}_1 \cdots \mathbf{u}_k]$. Then $X^*AX = \operatorname{diag}(\lambda_1(A), \ldots, \lambda_k(A))$ so that

$$\lambda_k(A) \leq \max\{\lambda_k(X^*AX) : X \in M_{n,k}, X^*X = I_k\}.$$

Conversely, suppose X has orthonomals column $\mathbf{x}_1, \ldots, \mathbf{x}_k$ spanning a subspace V_1. Let $\mathbf{u}_k, \ldots, \mathbf{u}_n$ span a subspace V_2 of dimension $n-k+1$. Then there is a unit vector $\mathbf{v} = \sum_{j=1}^k x_j\mathbf{x}_j = \sum_{j=k}^n y_j\mathbf{u}_j$. Let $\mathbf{x} = (x_1, \ldots, x_k)^t, \mathbf{y} = (y_k, \ldots, y_{n-k})^t$, $Y = [\mathbf{u}_k \ \ldots \ \mathbf{u}_{k+1}]$. Then $\mathbf{v} = X\mathbf{x} = Y\mathbf{y}$ so that $Y^*AY = \operatorname{diag}(\lambda_k(A), \ldots, \lambda_n(A))$. By Rayleigh principle,

$$\lambda_k(X^*AX) \leq \mathbf{x}^*X^*AX\mathbf{x} = \mathbf{y}^*Y^*AY\mathbf{y} \leq \lambda_k(A).$$

□

4.3 Change of eigenvalues under perturbation

Theorem 4.3.1 *Suppose $A, B \in M_n$ are Hermitian such that $A \geq B$. Then $\lambda_k(A) \geq \lambda_k(B)$ for all $k = 1, \ldots, n$.*

Proof. Let $A = B + P$, where P is positive semidefinite. Suppose $k \in \{1, \ldots, n\}$. There is $Y \in M_{n,k}$ with $Y^*Y = I_k$ such that

$$\lambda_k(B) = \lambda_k(Y^*BY) = \max\{\lambda_k(X^*BX) : X \in M_{m,n}, X^*X = I_k\}.$$

Let $\mathbf{y} \in \mathbb{C}^k$ be a unit eigenvector of Y^*AY corresponding to $\lambda_k(X^*AX)$. Then

$$\begin{aligned} \lambda_k(A) &= \max\{\lambda_k(X^*AX) : X \in M_{m,n}, X^*X = I_k\} \\ &\geq \lambda_k(Y^*AY) = \mathbf{y}^*Y^*(B+P)Y\mathbf{y} = \mathbf{y}^*Y^*BY\mathbf{y} + \mathbf{y}^*Y^*PY\mathbf{y} \\ &\geq \mathbf{y}^*Y^*BY\mathbf{y} \geq \lambda_k(Y^*BY) = \lambda_k(B). \end{aligned}$$

□

Theorem 4.3.2 (Lidskii) *Let $A, B, C = A+B \in M_n$ be Hermitian matrices with eigenvalues $a_1 \geq \cdots \geq a_n$, $b_1 \geq \cdots \geq b_n$, $c_1 \geq \cdots \geq c_n$, respectively. Then $\sum_{j=1}^n a_j + \sum_{j=1}^n b_j = \sum_{j=1}^n c_j$ and for any $1 \leq r_1 < \cdots < r_k \leq n$,*

$$\sum_{j=1}^k b_{n-j+1} \leq \sum_{j=1}^k (c_{r_j} - a_{r_j}) \leq \sum_{j=1}^k b_j.$$

Proof. Suppose $1 \leq r_1 < \cdots < r_k \leq n$. We want to show $\sum_{j=1}^k (c_{r_j} - a_{r_j}) \leq \sum_{j=1}^k b_j$. Replace B by $B - b_k I$. Then each eieganvelue of B and each eigenvalue of $C = A + B$ will be changed by $-b_k$. So, it will not affect the inequalities. Suppose $B = \sum_{j=1}^n b_j \mathbf{x}_j \mathbf{x}_j^*$. Let $B_+ = \sum_{j=1}^k b_j \mathbf{x}_j \mathbf{x}_j^*$. Then

$$\begin{aligned} \sum_{j=1}^k (c_{r_j} - a_{r_j}) &\leq \sum_{j=1}^k (\lambda_{r_j}(A + B_+) - \lambda_{r_j}(A)) \quad \text{because } \lambda_j(A+B) \leq \lambda_j(A+B_+) \text{ for all } j \\ &\leq \sum_{j=1}^n (\lambda_j(A + B_+) - \lambda_j(A)) \quad \text{because } \lambda_j(A) \leq \lambda_j(A+B_+) \text{ for all } j \\ &= \operatorname{tr}(A + B_+) - \operatorname{tr}(A) = \sum_{j=1}^k \lambda_j(B_+) = \sum_{j=1}^k b_j. \end{aligned}$$

Replacing (A, B, C) by $(-A, -B, -C)$, we get the other inequalities. □

Lemma 4.3.3 *Suppose $A \in M_{m,n}$ has nonzero singular values $s_1 \geq \cdots \geq s_k$. Then $\begin{pmatrix} 0_m & A \\ A^* & 0_n \end{pmatrix}$ has nonzero eigenvalues $\pm s_1, \ldots, \pm s_k$.*

Theorem 4.3.4 *Let $A, B, C \in M_{m,n}$ with singular values $a_1 \geq \cdots \geq a_n, b_1 \geq \cdots \geq b_n$ and $c_1, \ldots, c_n$, respectively. Then for any $1 \leq j_1 < \cdots < j_k \leq n$, we have*

$$\sum_{j=1}^k (c_{r_j} - a_{r_j}) \leq \sum_{j=1}^k b_j.$$

4.4 Eigenvalues of principal submatrices

Theorem 4.4.1 *There is a positive matrix* $C = \begin{pmatrix} A & * \\ * & B \end{pmatrix}$ *with* $A \in M_k$ *so that* A, B, C *have eigenvalues* $a_1 \geq \cdots \geq a_k$, $b_1 \geq \cdots \geq b_{n-k}$ *and* $c_1 \geq \cdots \geq c_n$, *respectively, if and only if there are positive semi-definite matrices* $\tilde{A}, \tilde{B}, \tilde{C} = \tilde{A} + \tilde{B}$ *with eigenvalues* $a_1 \geq \cdots \geq a_k \geq 0 = a_{k+1} = \cdots = a_n$, $b_1 \geq \cdots \geq b_{n-k} \geq 0 = b_{n-k+1} = \cdots = b_n$, *and* $c_1 \geq \cdots \geq c_n$.

Consequently, for any $1 \leq j_1 < \cdots < j_k \leq n$, *we have* $\sum_{j=1}^{k}(c_{r_j} - a_{r_j}) \leq \sum_{j=1}^{k} b_j$.

Proof. To prove the necessity, let $C = \hat{C}^*\hat{C}$ with $\hat{C} = [C_1\ C_2] \in M_n$ with $C_1 \in M_{n,k}$. Then $A = C_1^*C_1$ has eigenvalues $a_1, \ldots, a_k$, and $B = C_2^*C_2$ has eigenvalues $b_1, \ldots, b_{n-k}$. Now, $\tilde{C} = \hat{C}\hat{C}^* = C_1C_1^* + C_2C_2^*$ also eigenvalues $c_1, \ldots, c_n$, and $\tilde{A} = C_1C_1^*, \tilde{B} = C_2C_2^*$ have the desired eigenvalues.

Conversely, suppose the $\tilde{A}, \tilde{B}, \tilde{C}$ have the said eigenvalues. Let $\tilde{A} = C_1C_1^*$, $\tilde{B} = C_2C_2^*$ for some $C_1 \in M_{n,k}, C_2 \in M_{n,n-k}$. Then $C = [C_1\ C_2]^* = [C_1\ C_2]$ have the desired principal submatrices. □

By the above theorem, one can apply the inequalities governing the eigenvalues of $\tilde{A}, \tilde{B}, \tilde{C} = \tilde{A} + \tilde{B}$ to deduce inequalities relating the eigenvalues of a positive semidefinite matrix C and its complementary principal submatrices. One can also consider general Hermitian matrix by studying $C - \lambda_n(C)I$.

Theorem 4.4.2 *There is a Hermitian (real symmetric) matrix* $C \in M_n$ *with principal submatrix* $A \in M_m$ *such that* C *and* A *have eigenvalues* $c_1 \geq \cdots \geq c_n$ *and* $a_1 \geq \cdots \geq a_m$, *respectively, if and only if*

$$c_j \geq a_j \quad \text{and} \quad a_{m-j+1} \geq c_{n-j+1}, \qquad j = 1, \ldots, m.$$

Proof. To prove the necessity, we may replace C by $C - \lambda_n(C)I$ and assume that C is positive semidefinite. Then by the previous theorem,

$$c_j - a_j \geq b_{n-m} \geq 0, \qquad j = 1, \ldots, m.$$

Applying the argument to $-C$, we get the conclusion.

To prove the sufficiency, we will construct $C - c_nI$ with principal submatrix $A - c_nI_m$. Thus, we may assume that all the eigenvalues involved are nonnegative.

We prove the converse by induction on $n - m \in \{1, \ldots, n-1\}$. Suppose $n - m = 1$.

We need only address the case $\mu_j \in (\lambda_{j+1}, \lambda_j)$ for $j = 1, \ldots, n-1$, since the general case $\mu_j \in [\lambda_{j+1}, \lambda_j]$ follows by a continuity argument. Alternatively, we can take away the pairs of $c_j = a_j$ or $a_j = c_{j+1}$ to get a smaller set of numbers that still satisfy the interlacing inequalities and apply the following arguments.

We will show how to choose a real orthogonal matrix Q such that $C = Q^t \text{diag}(c_1, \ldots, c_n) Q$ has the leading principal submatrix $A \in M_{n-1}$ with eigenvalues $a_1 \geq \cdots \geq a_{n-1}$. To this end, let Q have last column $u = (u_1, \ldots, u_n)^t$. By the adjoint formula for the inverse

$$[(zI - C)^{-1}]_{nn} = \frac{\det(zI_{n-1} - A))}{\det(I - C)} = \frac{\prod_{j=1}^{n-1}(z - a_j)}{\prod_{j=1}^{n}(z - c_j)},$$

but we also have the expression

$$(zI - A)^{-1}_{nn} = u^t(zI - \text{diag}(\lambda_1, \ldots, \lambda_n))^{-1} u = \sum_{i=1}^{n} \frac{u_i^2}{(z - c_i)}.$$

Equating these two, we see that $A(n)$ has characteristic polynomial $\prod_{i=1}^{n-1}(z - \mu_i)$ if and only if

$$\sum_{i=1}^{n} u_i^2 \prod_{j \neq i}(z - a_i) = \prod_{i=1}^{n-1}(z - c_i).$$

Both sides of this expression are polynomials of degree $n-1$ so they are identical if and only if they agree at the n distinct points $c_1, \ldots, c_n$, or equivalently,

$$u_k^2 = \frac{\prod_{j=1}^{n-1}(c_k - a_j)}{\prod_{j \neq k}(c_k - c_j)} \equiv w_k, \qquad k = 1, \ldots, n.$$

Since $(c_k - a_j)/(c_k - c_j) > 0$ for all $k \neq j$, we see that $w_k > 0$. Thus if we take $u_k = \sqrt{w_k}$ then A has eigenvalues $a_1, \ldots, a_{n-1}$.

Now, suppose $m < n - 1$. Let

$$\tilde{c}_j = \begin{cases} \max\{c_{j+1}, a_j\} & 1 \leq j \leq m, \\ \min\{c_j, a_{m-n+j+1}\} & m < j < n. \end{cases}$$

Then

$$c_1 \geq \tilde{c}_1 \geq c_2 \geq \cdots \geq c_{n-1} \geq \tilde{c}_{n-1} \geq c_n,$$

and

$$\tilde{c}_j \geq a_j \geq \tilde{c}_{n-m-1+j}, \quad j = 1, \ldots, m.$$

By the induction assumption, we can construct a Hermitian $\tilde{C} \in M_{n-1}$ with eigenvalues $\tilde{c}_1 \geq \cdots \geq \tilde{c}_{n-1}$, whose $m \times m$ leading principal submatrix has eigenvalues $a_1 \geq \cdots \geq a_m$, and $\tilde{C}$ is the leading principal submatrix of the real symmetric matrix $C \in M_n$ such that C has eigenvalues $c_1 \geq \cdots \geq c_n$. □

4.5 Eigenvalues and Singular values

Theorem 4.5.1 *Let* $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in M_n$ *with* $A_{11} \in M_k$*. Then* $|\det(A_{11})| \le \prod_{j=1}^k s_j(A)$*. The equality holds if and only if* $A = A_{11} \oplus A_{22}$ *such that* A_{11} *has singular values* $s_1(A), \ldots, s_k(A)$*.*

Proof. Let $\mathcal{S}(s_1, \ldots, s_n)$ be the set of matrices in M_n with singular values $s_1 \ge \cdots \ge s_n$. Suppose $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \mathcal{S}(s_1, \ldots, s_n)$ with $A_{11} \in M_k$ such that $|\det(A_{11})|$ attains the maximum value. We show that $A = A_{11} \oplus A_{22}$ and A_{11} has singular values $s_1 \ge \cdots \ge s_k$.

Suppose $U, V \in M_k$ are such that $U^* A_{11} V = \mathrm{diag}\,(\xi_1, \ldots, \xi_k)$ with $\xi_1 \ge \cdots \ge \xi_k \ge 0$. We may replace A by $(U^* \oplus I_{n-k})A(V \oplus I_{n-1})$ and assume that $A_{11} = \mathrm{diag}\,(\xi_1, \ldots, \xi_k)$.

Let $A = (a_{ij})$. We show that $A_{21} = 0$ as follows. Suppose there is a nonzero entry a_{s1} with $k < s \le n$. Then there is a unitary $X \in M_2$ such that $X \begin{pmatrix} a_{11} & a_{s1} \\ a_{1s} & a_{ss} \end{pmatrix}$ has $(1,1)$ entry equal to

$$\hat{\xi}_1 = \{|a_{11}|^2 + |a_{s1}|^2\}^{1/2} = \{\xi_1^2 + |a_{s1}|^2\}^{1/2} > \xi_1.$$

Let $\hat{X} \in M_n$ be obtained from I_n by replacing the submatrix in rows and columns $1, j$ by X. Then the leading $k \times k$ submatrix of $\hat{X}A$ is obtained from that of A by changing its first row from $(\xi_1, 0, \ldots, 0)$ to $(\hat{\xi}_1, *, \cdots, *)$, and has determinant $\hat{\xi}_1 \xi_2 \cdots \xi_k > \xi_1 \cdots \xi_k = \det(A_{11})$, contradicting the fact that $|\det(A_{11})|$ attains the maximum value. Thus, the first column of A_{21} is zero.

Next, suppose that there is $a_{s2} \ne 0$ for some $k < s \le n$. Then there is a unitary $X \in M_2$ such that $X \begin{pmatrix} a_{22} & a_{s2} \\ a_{2s} & a_{ss} \end{pmatrix}$ has $(1,1)$ entry equal to

$$\hat{\xi}_2 = \{|a_{22}|^2 + |a_{s2}|^2\}^{1/2} = \{\xi_2^2 + |a_{s2}|^2\}^{1/2} > \xi_2.$$

Then the leading $k \times k$ submatrix of $\hat{X}A$ is obtained from that of A by changing its first row from $(0, \xi_2, 0, \ldots, 0)$ to $(0, \hat{\xi}_2, *, \cdots, *)$, and has determinant $\xi_1 \hat{\xi}_2 \cdots \xi_k > \xi_1 \cdots \xi_k = \det(A_{11})$, which is a contradiction. So, the second column of A_{21} is zero. Repeating this argument, we see that $A_{21} = 0$.

Now, the leading $k \times k$ submatrix of $A^t \in \mathcal{S}(s_1, \ldots, s_n)$ also attains the maximum. Applying the above argument, we see that $A_{12}^t = 0$. So, $A = A_{11} \oplus A_{22}$.

Let $\hat{U}, \hat{V} \in M_{n-k}$ be unitary such that $\hat{U}^* A_{22} \hat{V} = \mathrm{diag}\,(\xi_{k+1}, \ldots, \xi_n)$. We may replace A by $(I_k \oplus \hat{U}^*)A(I_k \oplus \hat{V})$ so that $A = \mathrm{diag}\,(\xi_1, \ldots, \xi_n)$. Clearly, $\xi_k \ge \xi_{k+1}$. Otherwise, we may interchange kth and $(k+1)$st rows and also the columns so that the leading $k \times k$ submatrix of the resulting matrix becomes $\mathrm{diag}\,(\xi_1, \ldots, \xi_{k-1}, \xi_{k+1})$ with determinant larger than $\det(A_{11})$. So, $\xi_1, \ldots, \xi_k$ are the k largest singular values of A. □

Theorem 4.5.2 *Let $a_1, \ldots, a_n$ be complex numbers be such that $|a_1| \ge \cdots \ge |a_n|$ and $s_1 \ge \cdots \ge s_n \ge 0$. Then there is $A \in M_n$ with eigenvalues $a_1, \ldots, a_n$ and singular values $s_1, \cdots, s_n$ if and only if*

$$\prod_{j=1}^{n} |a_j| = \prod_{j=1}^{n} s_j, \quad \text{and} \quad \prod_{j=1}^{k} |a_j| \le \prod_{j=1}^{k} s_j \quad \text{for } j = 1, \ldots, n-1.$$

Proof. Suppose A has eigenvalues $a_1, \ldots, a_n$ and singular values $s_1 \ge \cdots \ge s_n \ge 0$. We may apply a unitary similarity to A and assume that A is in upper triangular form with diagonal entries $a_1, \ldots, a_n$. By the previous theorem, if A_k is the leading $k \times k$ submatrix of A, then $|a_1 \cdots a_k| = |\det(A_k)| \le \prod_{j=1}^k s_k$ for $k = 1, \ldots, n-1$, and $|\det(A)| = |a_1 \cdots a_n| = s_1 \cdots s_n$.

To prove the converse, suppose the asserted inequalities and equality on $a_1, \ldots, a_n$ and $s_1, \ldots, s_n$ hold. We show by induction that there is an upper triangular matrix $A = (a_{ij})$ with singular values $s_1 \ge \cdots \ge s_n$ and diagonal values $|a_1|, \ldots, |a_n|$. Then there will be a diagonal unitary matrix D such that DA has the desired eigenvalues and singular values. For notation simplicity, we assume $a_j = |a_j|$ in the following.

Suppose $n = 2$. Then $a_1 \le s_1$, and $a_1 a_2 = s_1 s_2$ so that $s_1 \ge a_1 \ge a_2 \ge s_2$. Consider

$$A(\theta, \phi) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} s_1 & \\ & s_2 \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}.$$

There is $\phi \in [0, \pi/2]$ such that the $(s_1 \cos\phi, s_2 \sin\phi)^t$ has norm $a_1 \in [s_2, s_1]$. Then we can find $\theta \in [0, \pi/2]$ such that $(\cos\theta, \sin\theta)(s_1 \cos\phi, s_2 \sin\phi) = a_1$. Thus, the first column of $A(\theta, \phi)$ equals $(a_1, 0)^t$, and $A(\theta, \phi)$ has the desired eigenvalues and singular values.

Suppose the result holds for matrices of size at most $n - 1 \ge 2$. Consider $(a_1, \ldots, a_n)$ and $(s_1, \ldots, s_n)$ satisfying the product equality and inequalities.

If $a_1 = 0$, then $s_n = 0$ and $A = s_1 E_{12} + \cdots + s_{n-1} E_{n-1,n}$ has the desired eigenvalues and singular values.

Suppose $a_1 > 0$. Let k be the maximum integer such that $s_k \ge a_1$. Then there is $A_1 = \begin{pmatrix} a_1 & * \\ 0 & \tilde{s}_{k+1} \end{pmatrix}$ with $\tilde{s}_{k+1} = s_k s_{k+1}/a_1 \in [s_{k-1}, s_{k+1}]$. Let

$$(\tilde{s}_1, \ldots, \tilde{s}_{n-1}) = (s_1, \ldots, s_{k-1}, \tilde{s}_{k+1}, s_{k+2}, \ldots, s_n).$$

We claim that $(a_2, \ldots, a_n)$ and $(\tilde{s}_1, \ldots, \tilde{s}_{n-1})$ satisfy the product equality and inequalities. First, $\prod_{j=2}^n a_j = \prod_{j=1}^n s_j / a_1 = \prod_{j=1}^{n-1} \tilde{s}_j$. For $\ell < k$,

$$\prod_{j=2}^{\ell} a_j \le \prod_{j=1}^{\ell-1} a_j \le \prod_{j=1}^{\ell-1} s_j = \prod_{j=1}^{\ell-1} \tilde{s}_j.$$

For $\ell \geq k+1$,

$$\prod_{j=2}^{\ell} a_j \leq \prod_{j=1}^{\ell} s_j / a_1 = \prod_{j=1}^{\ell-1} \tilde{s}_j.$$

So, there is $A_2 \in U\tilde{D}V^*$ in triangular form with diagonal entries $a_2, \ldots, a_n$, where $U, V \in M_{n-1}$ are unitary, and $\tilde{D} = \mathrm{diag}\,(\tilde{s}_1, \ldots, \tilde{s}_{n-1})$. Let

$$A = \begin{pmatrix} 1 & \\ & U \end{pmatrix} \begin{pmatrix} A_0 & \\ & \tilde{D} \end{pmatrix} \begin{pmatrix} 1 & \\ & V^* \end{pmatrix}$$

is in upper triangular form with diagonal entries $a_1, \ldots, a_n$ and singular values $s_1, \ldots, s_n$ as desired. □

4.6 Diagonal entries and singular values

Theorem 4.6.1 *Let $A \in M_n$ have diagonal entries $d_1, \ldots, d_n$ such that $|d_1| \geq \cdots \geq |d_n|$ and singular values $s_1 \geq \cdots \geq s_n$.*

(a) *For any $1 \leq k \leq n$, we have $\sum_{j=1}^{k} |d_j| \leq \sum_{j=1}^{k} s_j$. The equality holds if and only if there is a diagonal unitary matrix D such that $DA = A_{11} \oplus A_{22}$ such that A_{11} is positive semidefinite with eigenvalues $s_1 \geq \cdots \geq s_k$.*

(b) *We have $\sum_{j=1}^{n-1} |d_j| - |d_n| \leq \sum_{j=1}^{n-1} s_j - s_n$. The equality holds if and only if there is a diagonal unitary matrix D such that $DA = (a_{ij})$ is Hermitian with eigenvalues $s_1, \ldots, s_{n-1}, -s_n$ and $a_{nn} \leq 0$.*

Proof. (a) Let $\mathcal{S}(s_1, \ldots, s_n)$ be the set of matrices in M_n with singular values $s_1 \geq \cdots \geq s_n$. Suppose $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \mathcal{S}(s_1, \ldots, s_n)$ with $A_{11} \in M_k$ such that $|a_{11}| + \cdots + |a_{kk}|$ attains the maximum value. We may replace A by DA by a suitable diagonal unitary $D \in M_n$ and assume that $a_{jj} = |a_{jj}|$ for all $j = 1, \ldots, n$. If $a_{ij} \neq 0$ for any $j > k \geq i$, then there is a unitary $X \in M_2$ such that $X \begin{pmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{pmatrix}$ has $(1,1)$ entry equal to

$$\tilde{a}_{ii} = \{|a_{ii}|^2 + |a_{ji}|^2\}^{1/2} > |a_{ii}|.$$

Let $\hat{X} \in M_n$ be obtained from I_n by replacing the submatrix in rows and columns i, j by X. Then diagonal entries of the leading $k \times k$ submatrix $\hat{A}_{11}$ of $\hat{X}A$ is obtained from that of A by changing its (i,i) entry a_{ii} to $\hat{a}_{ii}$ so that $tr\,\hat{A}_{11} > \mathrm{tr}\,A_{11}$, which is a contradiction. So, $A_{12} = 0$. Applying the same argument to A^t, we see that $A_{12} = 0$. Now, A_{11} has singular values $\xi_1 \geq \cdots \geq \xi_k$. Then $A_{11} = PV$ for some positive semidefinite matrix P with eigenvalues $\xi_1, \ldots, \xi_k$ and a unitary matrix $V \in M_k$. Suppose $V = U\hat{D}U^*$ for some diagonal unitary $\hat{D} \in M_k$ and unitary $U \in M_k$. Then

$$\mathrm{tr}\, A_{11} = \mathrm{tr}\,(PU\hat{D}U^*) = \mathrm{tr}\, U^*PU\hat{D} \le \mathrm{tr}\, U^*PU = \mathrm{tr}\, P,$$

where the equality holds if and only if $\hat{D} = I_k$, i.e., $A_{11} = P$ is positive semidefinite. In particular, we can choose $B = \mathrm{diag}\,(s_1, \ldots, s_n)$ so that the sum of the k diagonal entries is $\sum_{j=1}^k s_j \ge \sum_{j=1}^k \xi_j = \mathrm{tr}\, A_{11}$. Thus, the eigenvalues of A_{11} must be $s_1, \ldots, s_k$ as asserted.

(b) Let $A = (a_{ij}) \in \mathcal{S}(s_1, \ldots, s_n)$ attains the maximum values $\sum_{j=1}^{n-1} |a_{jj}| - |a_{nn}|$. We may replace A by a diagonal unitary matrix and assume that $a_{ii} \ge 0$ for $j = 1, \ldots, n-1$, and $a_{nn} \le 0$. Let $A_{11} \in M_{n-1}$ be the leading $(n-1) \times (n-1)$ principal submatrix of A. By part (a), we may assume that A_{11} is positive semidefinite so that its trace equals to the sum of its singular values. Otherwise, there are $U, V \in M_{n-1}$ such that $U^*A_{11}V = \mathrm{diag}\,(\xi_1, \ldots, \xi_{n-1})$ with $\xi_1 + \cdots + \xi_{n-1} > \sum_{j=1}^{n-1} a_{jj}$. As a result, $(U^* \oplus [1])A(V \oplus [1]) \in \mathcal{S}(s_1, \ldots, s_n)$ has diagonal entries $\hat{d}_1, \ldots, \hat{d}_{n-1}, a_{nn}$ such that

$$\sum_{j=1}^{n-1} \hat{d}_j - |a_{nn}| > \sum_{j=1}^{n-1} a_{jj} - |a_{nn}|,$$

which is a contradiction.

Next, for $j = 1, \ldots, n-1$, let $B_j = \begin{pmatrix} a_{jj} & a_{jn} \\ a_{nj} & a_{nn} \end{pmatrix}$. We show that $|a_{jj}| - |a_{nn}| = s_1(B_j) - s_2(B_j)$ and B_j is Hermitian in the following. Note that $s_1(B_1)^2 + s_2(B_j)^2 = |a_{jj}|^2 + |a_{jn}|^2 + |a_{nj}|^2 + |a_{nn}|^2$ and $s_1(B_j)s_2(B_j) = |a_{jj}a_{nn} - a_{jn}a_{nj}|$ so that $-a_{jj}a_{nn} = |a_{jj}a_{nn}| \ge s_1(B_j)s_2(B_j) - |a_{jn}a_{nj}|$. Hence,

$$\begin{aligned}
(|a_{jj}| - |a_{nn}|)^2 &= (a_{jj} + a_{nn})^2 = a_{jj}^2 + a_{nn}^2 + 2a_{jj}a_{nn} \\
&\le s_1(B_j)^2 + s_2(B_j)^2 - (|a_{jk}|^2 + |a_{kj}|^2) - 2(s_1(B_j)s_2(B_j) - |a_{jn}a_{nj}|) \\
&= (s_1(B_j) - s_2(B_j))^2 - (|a_{jk}| - |a_{kj}|)^2 \\
&\le (s_1(B_j) - s_2(B_j))^2.
\end{aligned}$$

Here the two inequalities become equalities if and only if $|a_{jk}| = |a_{kj}|$ and $|a_{jn}a_{nj}| = a_{jn}a_{nj}$, i.e., $a_{jn} = \bar{a}_{nj}$ and B_j is Hermitian.

By the above analysis, $|a_{jj}| - |a_{nn}| \le s_1(B_j) - s_2(B_j)$. If the inequality is strict, there are unitary $X, Y \in M_2$ such that $X^*B_jY = \mathrm{diag}\,(s_1(B_j), s_2(B_j))$. Let $\hat{X}$ be obtained from I_n by replacing the 2×2 submatrix in rows and columns j, n by X. Similarly, we can construct $\hat{Y}$. Then $\hat{X}, \hat{Y} \in M_n$ are unitary and $\hat{X}^*A\hat{Y}$ has diagonal entries $\hat{d}_1, \ldots, \hat{d}_n$ obtained from that of A by changing (a_{jj}, a_{nn}) to $(s_1(B_j), s_2(B_j))$. As a result,

$$\sum_{j=1}^{n-1} \hat{d}_j - |\hat{d}_n| > \sum_{j=1}^{n-1} a_{jj} - |a_{nn}|,$$

which is a contradiction. So, B_j is Hermitian for $j = 1, \dots, n-1$. Hence, A is Hermitian, and

$$\mathrm{tr}\, A = a_{11} + \cdots + a_{nn} = a_{11} + \cdots + a_{n-1,n-1} - a_{nn}.$$

Suppose A has eigenvalues $\lambda_1, \dots, \lambda_n$ with $|\lambda_j| = s_j(A)$ for $j = 1, \dots, n$. Because $0 \geq a_{nn} \geq \lambda_n$, we see that $\mathrm{tr}\, A = \sum_{j=1} \lambda_j \leq \sum_{j=1}^{n-1} s_j - s_n$. Clearly, the equality holds. Else, we have $B = \mathrm{diag}\,(s_1, \dots, s_n) \in \mathcal{S}(s_1, \dots, s_n)$ attaining $\sum_{j=1}^{n-1} s_j - s_n$. The result follows. □

Recall that for two real vectors $\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n)$, we say that $\mathbf{x} \prec_w \mathbf{y}$ is the sum of the k largest entries of $\mathbf{x}$ is not larger than that of $\mathbf{y}$ for $k = 1, \dots, n$.

Theorem 4.6.2 *Let $d_1, \dots, d_n$ be complex numbers such that $|d_1| \geq \cdots \geq |d_n|$. Then there is $A \in M_n$ with diagonal entries $d_1, \dots, d_n$ and singular values $s_1 \geq \cdots \geq s_n$ if and only if*

$$(|d_1|, \dots, |d_n|) \prec_w (s_1, \dots, s_n) \qquad \text{and} \qquad \sum_{j=1}^{n-1} |d_j| - |d_n| \leq \sum_{j=1}^{n-1} s_j - s_n.$$

Proof. The necessity follows from the previous theorem. We prove the converse by induction on $n \geq 2$. We will focus on the construction of A with singular values $s_1, \dots, s_n$, and diagonal entries $d_1, \dots, d_{n-1}, d_n$ with $d_1, \dots, d_n \geq 0$.

Suppose $n = 2$. We have $d_1 + d_2 \leq s_1 + s_2, d_1 - d_2 \leq s_1 - s_2$. Let $A = \begin{pmatrix} d_1 & a \\ -b & d_2 \end{pmatrix}$ such that $a, b \geq 0$ satisfies $ab = s_1 s_2 - d_1 d_2$ and $a^2 + b^2 = s_1^2 + s_2^2 - d_1^2 - d_2^2$. Such a, b exist because

$$2(s_1 s_2 - d_1 d_2) = 2ab \leq a^2 + b^2 = s_1^2 + s_2^2 - d_1^2 - d_2^2.$$

Suppose the result holds for matrices of sizes up to $n - 1 \geq 2$. Consider $(d_1, \dots, d_n)$ and $(s_1, \dots, s_n)$ that satisfy the inequalities. Let k be the largest integer k such that $s_k \geq d_1$.

If $k \leq n-2$, there is $B = \begin{pmatrix} d_1 & * \\ * & \hat{s} \end{pmatrix}$ with singular values s_k, s_{k+1}, where $\hat{s} = s_k + s_{k+1} - d_1$. One can check that $(d_2, \dots, d_n)$ and $(s_1, \dots, s_{k-1}, \hat{s}, s_{k+2}, \dots, s_n)$ satisfy the inequalities for the $n-1$ case so that there are unitary $U, V \in M_{n-1}$ such that UDV^* has diagonal entries $d_2, \dots, d_n$, where $D = \mathrm{diag}\,(\hat{s}, s_1, \dots, s_{k-1}, s_{k+2}, \dots, s_n)$. Thus,

$$A = ([1] \oplus U)(B \oplus \mathrm{diag}\,(s_1, \dots, s_{k-1}, s_{k+2}, \dots, s_n)([1] \oplus V^*)$$

has diagonal entries $d_1, \dots, d_n$ and singular values $s_1, \dots, s_n$.

Now suppose $k \geq n-1$, let

$$\hat{s} = \max\left\{0, d_n + s_n - s_{n-1}, \sum_{j=1}^{n-1} d_j - \sum_{j=1}^{n-2} s_j\right\} \leq \min\left\{s_{n-1}, s_{n-1} + s_n - d_n, \sum_{j=1}^{n-2}(s_j - d_j) + d_{n-1}\right\}.$$

It follows that

$$(d_n, \hat{s}) \prec_w (s_{n-1}, s_n), \qquad |d_n - \hat{s}| \le s_{n-1} - s_n,$$

$$(d_1, \ldots, d_{n-1}) \prec_w (s_1, \ldots, s_{n-2}, \hat{s}) \quad \text{and} \quad \sum_{j=1}^{n-2} d_j - d_{n-1} \le \sum_{j=1}^{n-2} s_j - \hat{s}.$$

So, there is $C \in M_2$ with singular values s_{n-1}, s_n and diagonal elements $\hat{s}, d_n$. Moreover, there are unitary matrix $X, Y \in M_{n-1}$ such that $X \operatorname{diag}(s_1, \ldots, s_{n-2}, \hat{s}) Y^*$ has diagonal entries $d_1, \ldots, d_{n-1}$. Thus,

$$A = (X \oplus [1])(\operatorname{diag}(s_1, \ldots, s_{n-2}) \oplus C)(Y^* \oplus [1])$$

will have the desired diagonal entries and singular values. □

4.7 Final remarks

The study of matrix inequalities has a long history and is still under active research. One of the most interesting question raised in 1960's and was finally solved in 2000's is the following.

Problem Determine the necessary and sufficient conditions for three set of real numbers $a_1 \ge \cdots \ge a_n, b_1 \ge \cdots \ge b_n, c_1 \ge \cdots \ge c_n$ for the existence of three (real symmetric) Hermitian matrices A, B and $C = A+B$ with these numbers as their eigenvalues, respectively.

It was proved that the conditions can be described in terms of the equality $\sum_{j=1}^n (a_j + b_j) = \sum_{j=1}^n c_j$ and a family of inequalities of the form

$$\sum_{j=1}^k (a_{u_j} + b_{v_j}) \ge \sum_{j=1}^k c_{w_j}$$

for certain subsequences $(u_1, \ldots, u_k), (v_1, \ldots, v_k), (w_1, \ldots, w_k)$ of $(1, \ldots, n)$.

There are different ways to specify the subsequences. A. Horn has the following recursive way to define the sequences.

1. If $k = 1$, then $w_1 = u_1 + v_1 - 1$. That is, we have $a_u + b_v \ge c_{u+v-1}$.
2. Suppose $k < n$ and all the subsequences of length up to $k-1$ are specified. Consider subsequences $(u_1, \ldots, u_k), (v_1, \ldots, v_k), (v_1, \ldots, v_k)$ satisfying $\sum_{j=1}^k (u_j + v_j) = \sum_{j=1}^k w_j + k(k+1)/2$, and for any lenth ℓ specified subsequences $(\alpha_1, \ldots, \alpha_\ell), (\beta_1, \ldots, \beta_\ell), (\gamma_1, \ldots, \gamma_\ell)$ of $(1, \ldots, n)$ with $\ell < k$,
$$\sum_{j=1}^{\ell} (u_{\alpha_j} + v_{\beta_j}) \ge \sum_{j=1} w_{\gamma_j}.$$

Consequently, the subsequences $(u_1,\ldots,u_k),(v_1,\ldots,v_k),(w_1,\ldots,w_k)$ of $(1,\ldots,n)$ is a Horn's sequence triples of length k if and only if there are Hermitian matrices $U,V,W=U+V$ with eigenvalues

$$u_1-1\le u_2-2\le\cdots\le u_k-k,\, v_1-1\le v_2-2\le\cdots\le v_k-k,\, w_1-1\le w_2-2\le\cdots\le w_k-k,$$

respectively. This is known as the saturation conjecture/theorem.

Special cases of the above inequalities includes the following inequalities of Thompson, which reduces to the Weyl's inequalities when $k=1$.

Theorem 4.7.1 *Suppose $A,B,C=A+B\in M_n$ are Hermitian matrices with eigenvalues $a_1\ge\cdots\ge a_n, b_1\ge\cdots b_n$ and $c_1\ge\cdots\ge c_n$, respectively. For any subequences $(u_1,\ldots,u_k),(v_1,\ldots,v_k)$, of $(1,\ldots,n)$, if $(w_1,\ldots,w_k)$ is such that $w_j=u_j+v_j-j\le n$ for all $j=1,\ldots,k$, then*

$$\sum_{j=1}^k (a_{u_j}+b_{v_j})\ge\sum_{j=1}^k c_{w_j}.$$

Proof. We prove the result by induction on n. Suppose $n=2$. If $k=n$ so that $(u_1,u_2)=(v_1,v_2)=(1,2)$, then the equality holds. If $k=1$, then $a_i+b_j\ge c_{i+j-1}$ for any $i+j\le 3$ by the Lidskii inequality.

Now, suppose the result holds for all matrices of size $n-1$. If $k=n$ so that $(u_1,\ldots,u_n)=(v_1,\ldots,v_n)$, then the equality holds. Suppose $k<n$. Let p be the largest integer such that $u_j=j$ for $j=1,\ldots,p$, and let q be the largest integer such that $v_j=j$ for $j=1,\ldots,q$. We may assume that $q\le p<n$. Else, interchange the roles of A and B.

Let $\{y_1,\ldots,y_n\}$ be an orthonormal set of eigenvectors of B and $\{z_1,\ldots,z_n\}$ be an orthonormal set of eigenvectors of C so that

$$By_j=a_jy_j,\qquad Cz_j=z_j,\qquad j=1,\ldots,n.$$

Suppose $Z\in M_{n,n-1}$ has orthonormal columns such that the column space of Z contains $z_1,\ldots,z_q,y_{q+2},\ldots,y_n$. Let $\tilde A=Z^*AZ,\tilde B=Z^*BZ,\tilde C=Z^*CZ$ have eigenvalues $\hat a_1\ge\cdots\ge\hat a_{n-1},\hat b_1\ge\cdots\ge\hat b_{n-1}$, and $\hat c_1\ge\cdots\ge\hat c_{n-1}$, respectively. By induction assumption,

$$\sum_{j=1}^q \hat c_{u_j+v_j-j}+\sum_{j=q+1}^k \hat c_{u_j+(v_j-1)-j}\le\sum_{j=1}^k\hat a_{u_j}+\sum_{j=1}^k\hat b_j+\sum_{j=q+1}^k b_{u_j+(v_j-1)-j}.$$

Because $u_j+v_j-j=j$ for $j=1,\ldots,q$, and the column space of Z contains $z_1,\ldots,z_q$, we see that $\hat c_j=c_j$ for $j=1,\ldots,q$. For $j=q+1,\ldots,k$, we have $c_{u_j+v_j-j}\le\hat c_{u_j+v_j-j-1}$, and hence

$$\sum_{j=1}^q c_{u_j+v_j-1}+\sum_{j=q+1}^k c_{u_j+v_j-j}\le\sum_{j=1}^q c_{u_j+v_j-1}+\sum_{j=q+1}^k\hat c_{u_j+v_j-j-1}.$$

Because $\hat{b}_j \le b_j$ for $j = 1, \dots, q$, and $\hat{b}_{u_j+v_j-j-1} = b_{u_j+v_j-j}$ for $j = q+1, \dots, k$ as the column spaces contains $y_{q+1}, \dots, y_n$, we have

$$\sum_{j=1}^{q} \hat{b}_j + \sum_{j=q+1}^{k} b_{u_j+(v_j-1)-j} \le \sum_{j=1}^{k} b_{u_j+v_j-j}.$$

The result follows. □

Applying the result to $-A-B = -C$, we see that for any subsequences $(u_1, \dots, u_k), (v_1, \dots, v_k)$ and $(w_1, \dots, w_k)$ with $w_j = u_i + v_j - j$ such that $u_k + v_k - k \le n$, we have

$$\sum_{j=1}^{k} (a_{n-u_j+1} + b_{n-v_j+1}) \le \sum_{j=1}^{k} (c_{n-w_j+1}).$$

Additional results and exercises

1. Suppose $n = 3$. List all the Horn sequences $(u_1, u_2), (v_1, v_2), (w_1, w_2)$ of length 2, and list all the Thompson standard sequences $(u_1, u_2), (v_1, v_2)$ and $(w_1, w_2) = (u_1 + v_1 - 1, u_2 + v_2 - 2)$.

2. Suppose $A, B, C = A + B \in M_n$ are Hermitian matrices have eigenvalues $a_1 \ge \cdots \ge a_n, b_1 \ge \cdots \ge b_n$ and $c_1 \ge \cdots \ge c_n$, respectively. Show that if $C = (c_{ij})$ then $\sum_{j=1}^{k} c_{jj} \le \sum_{j=1}^{k} (a_j + b_j)$; the equality holds if and only if $A = A_{11} \oplus A_{22}, B = B_{11} \oplus B_{22}$ with $A_{11}, B_{11} \in M_k$ such that A_{11} and B_{11} have eigenvalues $a_1 \ge cdots \ge a_k, b_1 \ge \cdots \ge b_k$, respectively.

3. (Weyl's inequalities.) Suppose $A, B, C = A + B \in M_n$ are Hermitian matrices. For any $u, v \in \{1, \dots, n\}$ with $u + v - 1 \le n$, show that $\lambda_u(A) + \lambda_v(B) \ge \lambda_{u+v-1}(A + B)$.

   Hint: By induction on $n \ge 2$. Check the case for $n = 2$. Assume the result hold for matrices of size $n - 1$. Assume $v \le u$. Let $\{z_1, \dots, z_n\}$ and $\{y_1, \dots, y_n\}$ be orthonormal sets such that $By_j = b_j y_j$ and $Cz_j = c_j z_j$ for $j = 1, \dots, n$. If $Z \in M_{n,n-1}$ with orthonormal columns such that the column space of Z contains $y_1, \dots, y_u$ and $z_{q+2}, \dots, z_n$. Argue that

   $$c_{u+v-1} = \lambda_{u+v-2}(Z^*CZ) \le \lambda_u(Z^*AZ) + \lambda_{v-1}(Z^*BZ) \le a_u + b_v.$$

4. Suppose $C = A + iB$ such that A and B has eigenvalues $a_1, \dots, a_n$ and $b_1, \dots, b_n$ such that $|a_1| \ge \cdots \ge |a_n|$ and $|b_1| \ge \cdots \ge |b_n|$. Show that if C has singular values $s_1, \dots, s_n$, then

   $$(a_1^2+b_n^2, \dots, a_n^2+b_1^2) \prec (s_1^2, \dots, s_n^2) \quad \text{and} \quad (s_1^2+s_n^2, \dots, s_n^2+s_1^2)/2 \prec (a_1^2+b_1^2, \dots, a_n^2+b_n^2).$$

   Hint: $2(A^2 + B^2) = CC^* + C^*C$.

5. Suppose $c_1 \geq a_1 \geq c_2 \geq a_2 \geq \cdots \geq a_{n-1} \geq c_n \geq a_n$ are $2n$ real numbers. Show that there is a nonnegative real vector $v \in \mathbb{R}^n$ such that $D + vv^t$ has eigenvalues $c_1 \geq \cdots \geq c_n$ for $D = \text{diag}\,(a_1, \ldots, a_n)$.

   Hint: Replace c_j by $c_j + \gamma$ and $a_j + \gamma$ for $j = 1, \ldots, n$, for a sufficiently large $\gamma > 0$, and assume that $c_n \geq a_n > 0$. By interlacing inequalities, there is $\tilde{C} = \begin{pmatrix} D & y \\ y^t & a \end{pmatrix}$. Show that $C = D + vv^t$ has eigenvalues $c_1 \geq \cdots \geq c_n$.

6. Suppose $A = \begin{pmatrix} \tilde{A} & * \\ 0 & * \end{pmatrix}$. Show that

$$s_1(A) \geq s_1(\tilde{A}) \geq s_2(A) \geq s_2(\tilde{A}) \geq \cdots \geq s_{n-1}(\tilde{A}) \geq s_n(A).$$

7. (Extra credit) Suppose $A, B \in M_n$. For any subsequences $(u_1, \ldots, u_k), (v_1, \ldots, v_k)$ and $(w_1, \ldots, w_k)$ of $(1, \ldots, n)$ such that $w_j = u_j + v_j - j$ for $j = 1, \ldots, k$, and $u_k + v_k - k \leq n$, we have

$$\prod_{j=1}^{k} s_{u_j}(A) s_{v_j}(B) \geq \prod_{j=1}^{k} s_{w_j}(AB).$$

   Hint: By induction on n. Check the case for $n = 2$. Assume that the result holds for matrices of size $n - 1$. If $k = n$, the equality holds. Suppose $k < n$. Let p be the largest integer such that $u_j = j$ for all $j = 1, \ldots, p$, and q be the largest integer such that $v_j = j$ for all $j = 1, \ldots, q$. We may assume that $q \leq p$. Let $C = AB$, $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ be orthonormal sets such that

$$B^* B u_j = s_j(B)^2 u_j \quad \text{and} \quad C^* C v_j = s_j(C)^2 v_j.$$

   Suppose U, V are unitary such that the first $n-1$ columns span a subspace containing $v_1, \ldots, v_1, u_{q+2}, \ldots, u_n$, and $V^* B U = \begin{pmatrix} \tilde{B} & * \\ 0 & * \end{pmatrix}$ with $\tilde{B} \in M_{n-1}$. Let W be unitary such that $W^* B V = \begin{pmatrix} \tilde{A} & * \\ 0 & * \end{pmatrix}$. Then $W^* A B V = \begin{pmatrix} \tilde{A}\tilde{B} & * \\ 0 & * \end{pmatrix}$. Apply induction assumption on $\tilde{A}\tilde{B}$ to finish the proof.

5 Norms

In many applications of matrix theory such as approximation theory, numerical analysis, quantum mechanics, one has to determine the "size" of a matrix, how near is one matrix to another, or how close is a matrix to a special class of matrices. We need concept of the norm (size) of a matrix. There are different ways to define the norm of a matrix, and the different definitions are useful in different applications.

5.1 Basic definitions and examples

Definition 5.1.1 *Let V be a linear space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A function $\nu : V \to [0, \infty)$ if*

(a) $\nu(v) \ge 0$ *for all $v \in V$; the equality holds if and only if $v = 0$.*

(b) $\nu(cv) = |c|\nu(v)$ *for any $c \in \mathbb{F}$ and $v \in V$.*

(c) $\nu(u + v) \le \nu(u) + \nu(v)$ *for all $u, v \in V$.*

Example 5.1.2 *Let $V = \mathbb{F}^n$. For $v = (v_1, \dots, v_n)^t \in \mathbb{F}^n$, let*

$$\ell_\infty(v) = \max\{|v_j| : 1, \dots, n\} \quad \textit{and} \quad \ell_p(v) = (\sum_{j=1}^n |v_j|^p)^{1/p} \quad \textit{for } p \ge 1$$

be the ℓ_∞ nrom and the ℓ_p norm.

Note that $\ell_2(v) = (\sum_{j=1}^n |v_j|^2)^{1/2}$ is the inner product norm.

For every $p \in [1, \infty]$, it is easy to verify (a) and (b). For $p = 1, \infty$, it is easy to verify the triangular inequality. For $p > 1$, the verification of $\ell_p(u + v) \le \ell_p(u) + \ell_p(v)$ is not so easy. We may change all the entries of u and v to their absolute values, and focus on vectors with nonnegative entries. to prove that the ℓ_p norm satisfies the triangle inequality $1 < p$, we establish the following.

Lemma 5.1.3 (Hölder's inequality) *Let $p, q > 1$ be such that $1/p + 1/q = 1$. For $u = (u_1, \dots, u_n)^t$ and $v = (v_1, \dots, v_n)^t$ with positive entries,*

$$\sum_{j=1} u_j v_j \le \ell_p(u)\ell_q(v).$$

The equality holds if and only if $(u_1^p, \dots, u_n^p)^t$ and $(v_1^q, \dots, v_n^q)^t$ are linearly dependent.

Proof. Replace (u, v) by $(u/\ell_p(u), v/\ell_q(v))$. We need to show that $u^t v \le 1$. Note that for two positive numbers a, b, we have

$$ab = \exp(\ln a + \ln b) = \exp((1/p)\ln(a^p) + (1/q)\ln(b^q))$$

$$\leq (1/p)\exp(\ln(a^p)) + (1/q)\exp(\ln(b^q)) = a^p/p + b^q/q,$$

where the equality holds if and only if $a^p = b^q$. Thus, we have $u_k v_k \leq u_k^p/p + v_k^q/q$, and

$$\sum_{j=1}^n u_j v_j \leq \ell_p(u)/p + \ell_q(v)/q = 1,$$

where the equality holds if and only if $u_j^p = v_j^q$ for all $j = 1, \dots, n$. □

Corollary 5.1.4 (Minkowski inequality) *Suppose* $p \in [1, \infty]$. *We have* $\ell_p(u+v) \leq \ell_p(u) + \ell_p(v)$.

Proof. The cases for $p = 1, \infty$ can be readily checked. Suppose $p > 1$. By the Hölder inequality, if $1 - 1/p = 1/q$, then

$$\begin{aligned} \sum_{j=1}^n (u_j + v_j)^p &= \sum_{j=1}^n u_j(u_j+v_j)^{p-1} + v_j(u_j+v_j)^{p-1} \\ &\leq \ell_p(u)(\sum_{j=1}^n (u_j+v_j)^p)^{1/q} + \ell_p(v)(\sum_{j=1}^n (u_j+v_j)^p)^{1/q} \quad \text{as } (p-1)q = p \\ &= (\ell_p(u) + \ell_p(v)) \left(\sum_{j=1}^n (u_j+v_j)^p\right)^{1/q}. \end{aligned}$$

Dividing both sides by $\left(\sum_{j=1}^n (u_j+v_j)^p\right)^{1/q}$, we get the conclusion. □

Next, we consider examples on matrices.

Example 5.1.5 Consider $V = M_{m,n}$. Using the inner product $\langle A, B\rangle = \mathrm{tr}\,(AB^*)$ on $M_{m,n}$, we have the inner product norm (a.k.a. Frobenius norm)

$$\|A\| = (\mathrm{tr}\, AA^*)^{1/2} = (\sum_{i,j} |a_{ij}|^2)^{1/2} = \sum_{j=1}^m s_j(A)^2.$$

One can define the $\ell_p(A) = (\sum_{i,j} |a_{ij}|^p)^{1/p}$, and define the Schatten p-norm by

$$S_p(A) = \ell_p(s(A)) = (\sum_{j=1}^m s_j(A)^p)^{1/p}.$$

The Schatten ∞-norm reduces to $s_1(A)$, which is also known as the spectral norm or operator norm defined by

$$\|A\| = \max\{\ell_2(Ax) : x \in \mathbb{C}^n, \ell_2(x) \leq 1\}.$$

When $m = n$, the Schatten 1-norm of A is just the sum of the singular values of A, and is also known as the trace norm.

One can also define the Ky Fan k-norm by $F_k(A) = \sum_{j=1}^k s_j(A)$ for $k = 1, \dots, m$.

Assertion *The Ky Fan k-norms and the Schatten p-norms satisfy the triangle inequalities.*

Proof. To prove the triangle inequality for the Ky Fan k-norm, note that if $C = A + B$, then $\begin{pmatrix} 0 & C \\ C^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} + \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}$. By the Lidskii inequalities $\sum_{j=1}^k s_j(C) \le \sum_{j=1}^k (s_j(A) + s_j(B))$. So, we have proved $s(C) \prec_w s(A) + s(B)$. It is easy so show that if $(c_1, \dots, c_m) \prec_w (\gamma_1, \dots, \gamma_m)$, then $\ell_p(c_1, \dots, c_m) \le \ell_p(\gamma_1, \dots, \gamma_n)$. Thus, we have

$$s_p(C) = \ell_p(s(C)) \le \ell_p(s(A) + s(B) \le \ell_p(s(A)) + \ell_p(s(B)) = S_p(A) + s_p(B).$$ □

For $A \in M_n$, one can define the numerical range and numerical radius of A by

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n,\ x^*x = 1\} \quad \text{and} \quad w(A) = \max\{|x^*Ax| : x \in \mathbb{C}^n,\ x^*x = 1\},$$

respectively. The spectral radius of $A \in M_n$ as

$$r(A) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}.$$

Example 5.1.6 *If $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, then*

$$\begin{aligned} W(A) &= \{(\bar{x}_1, \bar{x}_2)A(x_1, x_2)^t : |x_1|^2 + |x_2|^2 = 1\} = \{2\bar{x}_1 x_2 : |x_1|^2 + |x_2|^2 = 1\} \\ &= \{2\cos\theta \sin\theta e^{it} : \theta \in [0, \pi/2], t \in [0, 2\pi)\} = \{\mu \in \mathbb{C} : |\mu| \le 1\} \end{aligned}$$

Note that the numerical radius is a norm on M_n (homework), but the spectral radius is not.

Theorem 5.1.7 *Let $A \in M_n$. Then $W(A)$ is a compact convex set containing all the eigenvalues of A, and*

$$r(A) \le w(A) \le s_1(A) \le 2w(A).$$

Proof. Let $x, y \in \mathbb{C}^n$ be unit vectors, and $\alpha = x^*Ax, \beta = y^*Ay \in W(A)$. We need to show that the line segment joining α and β lies in $W(A)$. We assume $\alpha \ne \beta$ to avoid trivial consideration.

Note that $W(\xi A + \mu I) = \xi W(A) + \mu = \{\xi x^*Ax + \mu : x \in \mathbb{C}^n, x^*x = 1\}$. We may replace A by $B = (A - \alpha I)/(\beta - \alpha)$, and show that the line joining $x^*Bx = 0$ and $y^*By = 1$ lies in $W(B)$. We may further assume that $x^*By + y^*Bx \in \mathbb{R}$. Else, replace y by $e^{ir}y$ for a suitable $r \in [0, 2\pi)$.

Now, let $z(s) = [(1-s)x + sy]/\|(1-s)x + sy\|$ so that

$$z(s)^*Bz(s) = \frac{(1-s)^2x^*Bx + s(1-s)(x^*By + y^*Bx) + s^2y^*By}{\|(1-s)x + sy\|^s} \in W(B), \qquad s \in [0,1],$$

has real values vary from 0 to 1 continuously as s varies in $[0,1]$. So, $[0,1] \subseteq W(B)$.

The set $W(A)$ is compact means that it is bounded and contains all the boundary points. It follows from the fact that $W(A)$ is the image of the set of unit vectors in $\mathbb{C}^n$ under the continuous function $x \mapsto x^*Ax$.

Now, if λ is an eigenvalue of A, let x be a corresponding unit eigenvector of λ, then $x^*Ax = \lambda \in W(A)$. So, $r(A) \le w(A)$. Also, we have

$$w(A) = \max\{|x^*Ax| : x \in \mathbb{C}^n, x^*x = 1\} \le \max\{|x^*Ay| : x, y \in \mathbb{C}^n, x^*x = y^*y = 1\} \le s_1(A).$$

Finally, if $A = H + iG$ with $H = H^*, G = G^*$, then there are unit vectors $x, y \in \mathbb{C}^n$ such that

$$s_1(A) \le s_1(H + iG) \le s_1(H) + s_1(G) = |x^*Hx| + |y^*Gy| \le |x^*Ax| + |y^*Ay| \le 2w(A). \qquad \square$$

Definition 5.1.8 *A norm $\|\cdot\|$ on M_n is a matrix/algebra norm if*

$$\|AB\| \le \|A\|\|B\| \qquad \textit{for all } A, B \in M_n.$$

Suppose ν is a norm on $\mathbb{F}^n$. Then the operator norm induced by ν is defined by

$$\|A\|_\nu = \max\{\nu(Ax) : x \in \mathbb{C}^n, \nu(x) \le 1\}.$$

Note that every induced norm is a matrix norm. The Schatten p-norms, the Ky Fan k-norms, are matrix norms, but the numerical radius is not.

Example 5.1.9 *The operator norm induced by the ℓ_1-norm on $\mathbb{F}^n$ is the column sum norm defined by*

$$\|A\|_{\ell_1} = \max\{\sum_{j=1}^n : |a_{j\ell}| : \ell = 1, \ldots, n\}.$$

The operator norm induced by the ℓ_∞-norm on $\mathbb{F}^n$ is the row sum norm defined by

$$\|A\|_{\ell_\infty} = \max\{\sum_{j=1}^n : |a_{\ell j}| : \ell = 1, \ldots, n\}.$$

Theorem 5.1.10 *Let $A \in M_n$. Then $\lim_{k\to\infty} A^k = 0$ if and only if $r(A) < 1$.*

Proof. Let $A = S(J_1 \oplus \cdots \oplus J_k)S^{-1}$, where $J_1, \ldots, J_k$ are Jordan blocks. Assume $r(A) < 1$. We will show that $A^\ell \to 0$ as $\ell \to \infty$. It suffices to show that $J_i^\ell \to 0$ as $\ell \to \infty$ for each $i = 1, \ldots, k$.

Note that if μ satisfies $|\mu| < 1$ and $N_m = E_{12} + \cdots E_{m-1,m} \in M_m$, then for $\ell > m$,

$$(\mu I_m + N_m)^\ell = \sum_{j=0}^{m-1} \binom{\ell}{p} \mu^{\ell-p} N^p \to 0 \qquad \text{as} \quad \ell \to \infty$$

as $\lim_{\ell\to\infty} \binom{\ell}{p}\mu^{\ell-p} = 0$. Conversely, if $Ax = \mu x$ for some $|\mu| \geq 1$ and unit vector $x \in \mathbb{C}^n$, then $A^k x = \mu^k x$ so that $A^k \not\to 0$ as $k \to \infty$. □

Theorem 5.1.11 *Let $\|\cdot\|$ be a matrix norm on M_n. Then*

$$\lim_{k\to\infty} \|A^k\|^{1/k} = r(A).$$

Proof. Suppose μ is an eigenvalue of A such that $|\mu| = r(A)$. Let x be a unit vector such that $Ax = \mu x$. Then $|\mu^k| \|[x \cdots x]\| = \|A^k[x \cdots x]\| \leq \|A^k\| \|[x \cdots x]\|$. So, $|\mu^k| \leq \|A^k\|$.

Now, for any $\varepsilon > 0$, let $A_\varepsilon = A/(r(A) + \varepsilon)$. Then $\lim_{k\to\infty} A_\varepsilon^k = 0$. So, for sufficiently large $k \in \mathbb{N}$ we have $\|A^k/(rA + \varepsilon)^k\| < 1$. Hence, for any $\varepsilon > 0$, if k is sufficiently large, then

$$r(A) \leq \|A^k\|^{1/k} \leq r(A) + \varepsilon.$$

The result follows. □

Remark In the proof, we use the fact that the function $x \mapsto \|x\|$ is continuous. To see this, for any $\varepsilon > 0$, we can let $\delta = \varepsilon$, then $\|x - y\| < \delta$, we have $|\|x\| - \|y\|| \leq \|x - y\| = \delta = \varepsilon$.

Corollary 5.1.12 *Suppose $\|\cdot\|$ is a matrix norm on M_n such that $\|A\| \geq r(A)$ for all $A \in M_n$. If $\|A\| < 1$, then $A^k \to 0$ as $k \to \infty$.*

5.2 Geometric and analytic properties of norms

Let ν be a norm on a linear space V. Then

$$\mathcal{B}_\nu = \{x \in V : \nu(x) \leq 1\}$$

is the unit ball of the norm ν.

Theorem 5.2.1 *Let ν be a norm on a nonzero linear space V. Then $\mathcal{B}_v$ satisfies the following.*

(a) *The zero vector* 0 *is an interior point.*

(b) *For any* $\mu \in \mathbb{F}$ *with* $|\mu| = 1$,

$$\mathcal{B}_v = \mu\mathcal{B}_\nu = \{\mu x : x \in \mathcal{B}_\nu\}.$$

(c) *The set* $\mathcal{B}_\nu$ *is convex. That is if* $x, y \in \mathcal{B}_\nu$, *then* $tx + (1-t)y \in \mathcal{B}_\nu$.

Conversely, if V *is finite dimensional linear space over* $\mathbb{F}$ *and* $\mathcal{B}$ *is a set satisfying* (a) — (c)*, then we can define a norm* $\|\cdot\|$ *on* V *by* $\|x\| = 0$*, and for any nonzero* $x \in V$,

$$\|x\| = \sup\{t > 0 : x/t \in \mathcal{B}\} = \max\{t > 0 : x/t \in \mathcal{B}\}.$$

Theorem 5.2.2 *Suppose* ν_j *for* $j \in J$ *is a family of norm on a linear space* V *so that* 0 *is an interior point of* $\cap\mathcal{B}_{\nu_j}$. *Then* $\cap\mathcal{B}_{\nu_j}$ *is the unit norm ball of* ν *defined by*

$$\nu(x) = \sup\{\nu_j(x) : j \in J\}.$$

5.3 Inner product norm and the dual norm

Recall that for a linear space V, a scalar function on $V \times V$ is an inner product denoted by $\langle x, y\rangle \in \mathbb{F}$ if it satisfies

(a) $\langle x, x\rangle \geq 0$, where the equality holds if and only if $x = 0$,

(b) $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$,

(c) $\langle x, z\rangle = \overline{\langle z, x\rangle}$,

for any $a, b \in \mathbb{F}, x, y, z \in V$.

Theorem 5.3.1 *Suppose* V *is an inner product space. Then for any* $x, y \in V$,

$$\|x\| = \langle x, x\rangle^{1/2} \qquad x \in V$$

is a norm satisfying the Cauchy inequality

$$|\langle x, y\rangle| \leq \|x\|\|y\|$$

and the parallelogram identity

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Theorem 5.3.2 *Suppose* $\|\cdot\|$ *is a norm on a linear space* V *satisfying the parallelogram identity. Then one can define an inner product by* $\langle x, y\rangle = a + ib$ *with*

$$2a = \|x + y\|^2 - \|x\|^2 - \|y\|^2 \quad \text{and} \quad 2b = \|x + iy\|^2 - \|x\|^2 - \|y\|^2$$

such that $\|z\| = \langle z, z\rangle^{1/2}$ *for all* $z \in V$.

Remark 5.3.3 *Suppose V is an inner product space, and ν is a norm on V. One can define the dual norm on V by*

$$\nu^D(x) = \sup\{|\langle x, y\rangle| : \nu(y) \le 1\}.$$

We have $(\nu^D)^D = \nu$.

Example 5.3.4 *The dual norm of the ℓ_p norm on $\mathbb{F}^n$ is the ℓ_p norm with $1/p + 1/q = 1$.*

The dual norm of the Schatten p norm on $M_{m,n}$ is the Schatten q norm on $M_{m,n}$ with $1/p + 1/q = 1$.

The dual norm of the Ky Fan k-norm on $M_{m,n}$ with $m \ge n$ is $F_k^d(A) = \max\{\sum_{j=1}^n s_j(A), s_1(A)\}$

5.4 Symmetric norms and unitarily invariant norms

A norm on $\mathbb{F}^n$ is a symmetric norm if $\|x\| = \|Px\|$ for all permutation matrix P or diagonal unitary (orthogonal) matrix P.

A norm on $M_{m,n}(\mathbb{F})$ is a unitarily invariant norm (UI norm) if $\|UAV\| = \|A\|$ for any unitary $U \in M_m, V \in M_n$, and any $A \in M_{m,n}$.

Theorem 5.4.1 *Suppose $m \ge n$. Every UI norm $\|\cdot\|$ on $M_{m,n}$ corresponds to a symmetric norm ν on $\mathbb{R}^n$ such that*

$$\|A\| = \nu(s(A)) \qquad \text{for all } A \in M_{m,n}.$$

Proof. Suppose $\|\cdot\|$ is a UI norm. Then $\|A\| = \|\sum_{j=1}^n s_j(A)E_{jj}\|$ for any $A \in M_{m,n}$. Define $\nu : \mathbb{F}^n \to \mathbb{R}$ by $\nu(x) = \|\sum_{j=1}^n |x_j|E_{jj}\|$ for $x = (x_1, \dots, x_n)^t \in \mathbb{R}^n$. Then it is easy to verify that ν is a symmetric norm.

Conversely, if ν is a symmetric norm on $\mathbb{R}^n$, then define $\|\cdot\|$ by $\|A\| = \nu(s(A))$ for any $A \in M_{m,n}$. Then one can check that $\|A\|$ is a norm using the fact that $s(A+B) \prec s(A)+s(B)$ so that $\nu(s(A+B)) \le \nu(s(A)+s(B))$. □

Denote by GP_n the set of matrices equal to the product of a permutation matrix and a diagonal unitary (orthogonal) matrices if $\mathbb{F} = \mathbb{C}$ (if $\mathbb{F} = \mathbb{R}$). Let $c = (c_1, \dots, c_n) \in \mathbb{R}^n$ with $c_1 \ge \cdots \ge c_n \ge 0$. Define the c-norm on $\mathbb{F}^n$ by

$$\nu_c(x) = \max\{c^t Px : P \in GP_n\}$$

and the c-spectral norm on $M_{m,n}(\mathbb{F})$ by

$$\|A\|_c = \nu_c(s(A)).$$

If $c = (\underbrace{1, \dots, 1}_{k}, 0, \dots, 0)$, we get the $\nu_k(x)$ and the Ky Fan k-norm $F_k(A)$.

Lemma 5.4.2 *Suppose ν on $\mathbb{R}^n$ is a symmetric norm. Then for any $x \in \mathbb{R}^n$,*

$$\nu(x) = \max\{\nu_c(x) : c = (c_1, \dots, c_n), c_1 \geq \cdots \geq c_n, \nu^d(c) = 1\}.$$

Suppose $\|\cdot\|$ is a UI norm on $M_{m,n}(\mathbb{F})$. Then for any $A \in M_{m,n}$,

$$\|A\| = \max\{\|A\|_c : C = s(C) \text{ for some } C \in M_{m,n}, \|C\|^d = 1\}.$$

Theorem 5.4.3 *Let $x, y \in \mathbb{F}^n$. The following are equivalent.*

(a) $\nu_k(x) \leq \nu_k(y)$ *for all* $k = 1, \dots, n$.

(b) $\nu_c(x) \leq \nu_c(y)$ *for all nonzero* $c = (c_1, \dots, c_n)$ *with* $c_1 \geq \cdots \geq c_n \geq 0$.

(c) $\nu(x) \leq \nu(y)$ *for all symmetric norms* ν.

Proof. Suppose (a) holds. Then for any $c = (c_1, \dots, c_n)$ with $c_1, \dots, c_n$, if we set $d_n = c_n$ and $d_j = c_j - j + 1$ for $j = 1, \dots, n-1$, then $\nu_c(z) = \sum_{j=1}^n d_j \nu_j(z)$. Thus,

$$\nu_c(x) = \sum_{j=1}^n d_j \nu_j(x) \leq \sum_{j=1}^n d_j \nu_j(y) = \sum_{j=1}^n c_j y_j = \nu_c(y).$$

Suppose (b) holds. Let ν be a symmetric norm. Then for any $c = (c_1, \dots, c_n)$ with $c_1 \geq \cdots \geq c_n \geq 0$ with $\nu^d(c) = 1$, we have $\nu_c(x) \leq \nu_c(y)$. Thus, $\nu(x) = \nu(y)$.

The implication (b) $\Rightarrow$ (c) is clear. □

Theorem 5.4.4 *Let $A, B \in M_{m,n}(\mathbb{F}^n)$ with $m \geq n$. The following are equivalent.*

(a) $F_k(A) \leq F_k(B)$ *for all* $k = 1, \dots, n$.

(b) $\|A\|_c \leq \|B\|_c$ *for all nonzero* $c = (c_1, \dots, c_n)$ *with* $c_1 \geq \cdots \geq c_n \geq 0$.

(c) $\|A\| \leq \|B\|$ *for all UI norms* $\|\cdot\|$.

Proof. Similar to the last theorem. □

Theorem 5.4.5 *Let $\mathcal{R}_k \subseteq M_{m,n}$ be the set of matrices of rank at most k with $m \geq n > k$. Suppose $\|\cdot\|$ is a UI norm. If $A \in M_{m,n}$ such that $U^*AV = \sum_{j=1}^n s_j(A)E_{jj}$, then $A_k = U(\sum_{j=1}^k s_j(A)E_{jj})V^*$ satisfies*

$$\|A - A_k\| \leq \|A - X\| \qquad \text{for all } X \in \mathcal{R}_k.$$

Proof. Let $X \in \mathcal{R}_k$ and $C = A - X$. Then $s_j(X) = 0$ for $j > k$ so that

$$\sum_{j=1}^{\ell} s_{k+j}(A) = \sum_{j=1}^{\ell}(s_{k+j}(A) - s_{k+1}(X)) \le \sum_{j=1}^{\ell} s_j(C), \qquad \text{for all } \ell = 1, \ldots, n-k.$$

So, $(s_{k+1}(A), \ldots, s_n(A), 0, \ldots, 0) \prec_w s(C)$ and $\|A - A_k\| \le \|C\| = \|A - X\|$. □

Theorem 5.4.6 *Let $A \in M_n$ and $\|\cdot\|$ be a unitarily invariant norm.*

(a) $\|A - (A + A^*)/2\| \le \|A - H\|$ *for any* $H = H^* \in M_n$.

(b) $\|A - (A - A^*)/2\| \le \|A - iG\|$ *for any* $G = G^* \in M_n$.

Proof. (a) Let $H \in M_n$ be Hermitian, and let $A - H = \hat{H} + iG$. Suppose $Q \in M_n$ is unitary such that Q^*GQ is in diagonal form $g_1, \ldots, g_n$ such that $|g_1| \ge \cdots \ge |g_n|$. If $d_1, \ldots, d_n$ are the diagonal entries of $Q^*(H + iG)Q$, then

$$s(G) = (|g_1|, \ldots, |g_n|) \prec_w (|d_1|, \ldots, |d_n|) \prec_w (A - H).$$

Thus, $\|G\| = \|A - (A + A^*)/2\| \le \|A - H\|$.

(b) Similar to (a). □

Theorem 5.4.7 *Suppose $A, B \in M_n$ have singular values $a_1 \ge \cdots \ge a_n$ and $b_1 \ge \cdots \ge b_n$. Then for any UI norm $\|\cdot\|$,*

$$\|\sum_{j=1}^{m}(a_j + b_{n-j+1})E_{jj}\| \le \|A + B\| \le \|\sum_{j=1}^{m}(a_j + b_j)E_{jj}\|.$$

Proof. By Lidskii inequalities, for all $k = 1, \ldots, n$,

$$\sum_{j=1}^{k}[\lambda_j(A + B) - \lambda_j(A)] \le \sum_{j=1}^{k}\lambda_j(B) \quad \text{and} \quad \sum_{j=1}^{k}\lambda_{i_j}(A) - \lambda_{i_j}(-B) \le \sum_{j=1}^{k}\lambda_j(A - (-B)).$$

We get the majorization result. □

Theorem 5.4.8 *Let $\|\cdot\|$ be a UI norm on M_n.*

(a) *If P is positive semidefinite, then $\|P - I\| \le \|P - V\| \le \|P + V\|$ for any unitary $V \in M_n$.*

(b) *If $A = UP$ such that P is positive semidefinite and U is unitary, then*

$$\|A - U\| \le \|A - V\| \qquad \text{for any unitary } V \in M_n.$$

Proof. (a) Apply the previous theorem with $P = A$ and $B = I$.

(b) Use the fact that $\|A - V\| = \|UP - V\| = \|P - U^*V\| \ge \|P - I\| = \|UP - U\|$. □

5.5 Errors in computing inverse and solving linear equations

Theorem 5.5.1 *If $B \in M_n$ satisfies $r(B) < 1$, then $I - B$ is invertible and*

$$(I-B)^{-1} = \sum_{k=0}^{\infty} B^k.$$

Consequently, if $A \in M_n$ is invertible and E satisfies $r(A^{-1}E) < 1$, then $A+E$ is invertible and

$$A^{-1} - (A+E)^{-1} = \sum_{k=1}^{\infty} (A^{-1}E)^k A^{-1}.$$

Furthermore, if $\|\cdot\|$ is a matrix norm on M_n such that $\|A^{-1}E\| < 1$ and $\kappa(A) = \|A^{-1}\|\|A\|$, then

$$\frac{\|A^{-1} - (A+E)^{-1}\|}{\|A^{-1}\|} \le \frac{\|A^{-1}E\|}{1 - \|A^{-1}E\|} \le \frac{\kappa(A)}{1 - \kappa(A)(\|E\|/\|A\|)} \frac{\|E\|}{\|A\|}.$$

Proof. Use the identity $(I-A)(\sum_{j=1}^{k} A^j) = I - A^{k+1}$ and letting $k \to \infty$. □

The quantity $\kappa(A)$ is called the condition number of A with respect to the norm $\|\cdot\|$.

Important implication, the change of the inverse will affected by $\kappa(A)$. For example, if $\|A\| = s_1(A)$, and A is unitary, then

$$\frac{\|A^{-1} - (A+E)^{-1}\|}{\|A^{-1}\|} \le \frac{\|E\|}{\|A\| - \|E\|}.$$

So, the computation of A is very "stable".

We can apply the result to analysis the solution of $Ax = b$.

Corollary 5.5.2 *Let $A, E \in M_n$ and $x, b \in \mathbb{C}^n$ be such that $Ax = b$ and $(A+E)\hat{x} = b$. Suppose A and $(A+E)$ are invertible.*

$$x - \hat{x} = [A^{-1} - (A+E)^{-1}]b = [A^{-1} - (A+E)^{-1}]A^{-1}x.$$

Suppose $\|\cdot\|$ is a matrix norm on M_n such that $\|A^{-1}E\| < 1$, and if ν is a norm on $\mathbb{C}^n$ such that $\nu(Bz) \le \|B\|\nu(z)$ for all $B \in M_n$ and $z \in \mathbb{C}^n$. If $\kappa(A) = \|A^{-1}\|\|A\|$, then

$$\frac{\nu(x - \hat{x})}{\nu(x)} \le \frac{\|A^{-1}E\|}{1 - \|A^{-1}E\|} \le \frac{\kappa(A)}{1 - \kappa(A)(\|E\|/\|A\|)} \frac{\|E\|}{\|A\|}.$$

6 Additional topics

6.1 Location of eigenvalues

Theorem 6.1.1 (Gershgorin Theorem) *Let $A \in (a_{ij})$, and let*

$$G_j(A) = \{\mu \in \mathbb{C} : |\mu - a_{jj}| \le \sum_{i \ne j} |a_{ji}|\}.$$

Then the eigenvalues of A lies in $G(A) = \cup_{j=1}^n G_j(A)$. Furthermore, if $C = G_{i_1}(A) \cup \cdots \cup G_{i_j}(A)$ form a connected component of G, then C contains exactly k eigenvalues counting multipicities.

Proof. Suppose $Av = \lambda v$ with $v = (v_1, \ldots, v_n)$. Then for $i = 1, \ldots, n$,

$$\lambda v_i - a_{ii} v_i = \sum_{j \ne i} a_{ij} v_j.$$

Suppose v_i has the maximum size. Then

$$|\lambda - a_{ii}| = |\sum_{j \ne i} a_{ij} v_j / v_i| \le \sum_{j \ne i} |a_{ij}|.$$

To prove the last assertion. Let $A_t = D + t(A - D)$ with $D = \mathrm{diag}\,(a_{11}, \ldots, a_{nn})$. Then A_0 has eigenvalues $a_{11}, \ldots, a_{nn}$, and the eigenvalues and Gershgorin disk will change continuously according to $t \in [0, 1]$ until we get $A_1 = A$.

One can apply the result to A^t to get Gershgorin disks of different sizes centered at $a_{11}, \ldots, a_{nn}$. Also, one can apply the result to $S^{-1}AS$ for (simple) invertible S such that $G(S^{-1}AS)$ is small. In fact, if A is already in Jordan form, then for any $\varepsilon > 0$ there is S such that $S^{-1}AS$ has diagonal entries $\lambda_1, \ldots, \lambda_n$ and $(i, i+1)$ entries equal 0 or ε for $i = 1, \ldots, n-1$, and all other entries equal to 0. So, we have the following.

Theorem 6.1.2 *Let $A \in M_n$. Then*

$$\bigcap_{S \in M_n \text{ is invertible}} G(S^{-1}AS) = \{\lambda_1(A), \ldots, \lambda_n(A)\}.$$

One may use the Gershgorin theorem to study the zeros of a (monic) polynomial, namely, one can apply the result to the companion matrix C_f of $f(x)$ to get some estimate of the location of the zeros. One can further apply similarity to C_f to get better estimate for the zeros of $f(x)$.

6.2 Eigenvalues and principal minors

Theorem 6.2.1 *Let $A \in M_n$ with eigenvalues $\lambda_1, \dots, \lambda_n$. Then*

$$\det(zI - A) = (z - \lambda_1)\cdots(z - \lambda_n) = z^n - a_1 z^{n-1} + a_2 z^{n-2} - \cdots + (-1)^n a_n,$$

where for $m = 1, \dots, n$,

$$a_m = S_m(\lambda_1, \dots, \lambda_n) = \sum_{1 \le j_1 < \cdots < j_m \le n} (\lambda_{j_1} + \cdots + \lambda_{j_m})$$

is the sum of all $m \times m$ principal minors of A.

Proof. For any subseteq $J \subseteq \{1, \dots, n\}$, let $A[J]$ be the principal submatrix of A with row and column indices in J. Consider the expansion $\det(zI - A)$. The coefficient of z^{n-j} comes from the sum of the leading coefficients of $(-1)^j \det(A[J]) \det(zI - A[\overline{J}])$ for all different j-element subsets J of $\{1, \dots, n\}$. The result follows. □

6.3 Nonnegative Matrices

In this section, we consider positive (nonnegative) matrices A, i.e., the entries of A are positive (nonnegative) real numbers. Denote by $|A|$ the matrix obtained from A by changing its entries to their absolute values (norm). Similarly, we consider $|v|$ of a vector v.

Theorem 6.3.1 (Perron-Frobenius Theorem) *Suppose $A \in M_n$ is nonnegative such that A^k is positive for some positive integer k. Then the following holds.*

(a) *$r(A) > 0$ is an algebraically simple eigenvalue of A such that $r(A) > |\lambda|$ for all other eigenvalue λ of A.*

(b) *There is a unique positive vector x with $\ell_1(x) = 1$ such that $Ax = r(A)x$, and there is a unique positive vector y with $y^t x = 1$ and $y^t A = r(A) y^t$.*

(c) *Let x and y be the vectors in* (b). *Then $(r(A)^{-1}A)^m \to xy^t$ as $m \to \infty$.*

We first prove a lemma.

Lemma 6.3.2 *Suppose $A \in M_n$ is nonnegative with row sums $r_1, \dots, r_n$.*

(a) *For any nonnegative matrix P, $r(A) \le r(A + P)$.*

(b) *If all the row sums are the same, then $r(A) = r_1$. In general,*

$$\min\{r_i : 1 \le i \le n\} \le r(A) \le \max\{r_i : 1 \le i \le n\}.$$

Proof. (a) If $B = A + P$, then for any positive integer k, $B^k - A^k$ is nonnegative so that $\|A^k\|_{\ell_\infty} \le \|B^k\|_{\ell_\infty}$. Hence,

$$r(A) = \lim_{k\to\infty} \|A^k\|_{\ell_\infty}^{1/k} \le \lim_{k\to\infty} \|B^k\|_{\ell_\infty}^{1/k} = r(B).$$

(b) Suppose all the row sums are the same. Let $e = (1, \ldots, 1)^t$. Then $Ae = r_1 e$ so that r_1 is an eigenvalue. By Gershgorin Theorem all eigenvalues lie in

$$\bigcup_{i=1}^n \left\{\mu \in \mathbb{C} : |\mu - a_{ii}| \le \sum_{j\neq i} a_{ij}\right\}.$$

Thus, all eigenvalues lie in the set $\{\mu \in \mathbb{C} : |\mu| \le r_1\}$. Hence, $r_1 = r(A)$.

In general, let P be a nonnegative matrix such that $B = A + P$ has all row sum equal to $\|A\|_{\ell_\infty}$. Then $r(A) \le r(B) = \|A\|_{\ell_\infty}$.

Similarly, let Q be a nonnegative matrix such that $\hat{B} = A - Q$ is nonnegative with all row sum equal to $r_\ell = \min\{r_i : 1 \le i \le n\}$. Then $r_\ell = r(\hat{B}) \le r(A)$. □

Proof of Theorem 6.3.1. Assume $B = A^k$ is positive. Then $r(B)$ is larger than the minimum row sum of B so that $0 < r(B) = r(A)^k$. Note that Bv is positive for any nonzero vector $v \ge 0$.

Assertion 1 *Let λ be an eigenvalue of B. Either $|\lambda| < r(B)$ or $\lambda = r(B)$ with an eigenvector x such that $x = e^{i\theta}|x|$ for some $\theta \in \mathbb{R}$.*

Proof. Let λ be an eigenvalue of B such that $|\lambda| = r(B)$, and x be an eigenvector. Then $r(B)|x| = |r(B)x| = |Bx| \le B|x|$. We claim that the equality holds. If it is not true, we can set $z = B|x|$ so that $y = (B - r(B))|x| = z - r(B)|x| \neq 0$ is nonnegative. Then

$$0 < By = Bz - r(B)B|x| = Bz - r(B)z.$$

So, $z = (z_1, \ldots, z_n)^t$ has positive entries, and for $Z = \mathrm{diag}\,(z_1, \ldots, z_n)$, we have

$$Z^{-1}(BZe - r(B)Ze) = Z^{-1}BZe - r(B)e = Z^{-1}By > 0.$$

If follows that $Z^{-1}BZ$ has minimum row sum $r(B) + \delta$, where $\delta = \ell_\infty(Z^{-1}By) > 0$. So, $r(Z^{-1}BZ) \ge r(B) + \delta$, which is a contradiction.

Now, $r(B)|x| = B|x|$ has positive entries, and $|Bx| = r(B)|x| = B|x|$. Thus, $x = e^{i\theta}|x|$, i.e., x is the eigenspace of $r(B)$ and $\lambda = r(B)$. The proof of Assertion 1 is complete.

Assertion 2 *The value $r(B)$ is a simple eigenvalue of B with a unique positive positive eigenvector x satisfying $e^t x = 1$ and a unique positive left eigenvector y such that $y^t x = 1$. Moreover, there is an invertible matrix $S \in M_n$ such that x is the first column of S and y^t is the first row of S^{-1} satisfying $S^{-1}BS = [r(B)] \oplus B_1$ with $r(B_1) < r(B)$.*

Proof. Suppose $Bu = r(B)u$ and $Bv = r(B)v$ for two linearly independent vectors u and v such that $e^t|u| = e^t|v| = 1$. By the arguments in the previous paragraphs, we see that there are $\theta, \phi \in \mathbb{R}$ such that $u = e^{i\theta}|u|$ and $v = e^{i\phi}|v|$, such that $|u|, |v|$ have positive entries. So, there is $\beta > 0$ such that $|u| - \beta|v|$ is nonnegative with at least one zero entry. We have $r(B)(|u| - \beta|v|) = B(|u| - \beta|v|)$, and $B(|u| - \beta|v|)$ has a positive entries, which is a contradiction. So, $|u| = |v|$.

Let x be the unique positive eigenvector such that $Bx = r(B)x$ satisfying $e^t x = 1$. Then we can consider B^t and obtain a positive vector $B^t y = r(B)y$ satisfying $x^t y = 1$. Let $S = [x|S_1] \in M_n$ be such that the columns of $y^t S_1 = [0, \ldots, 0] \in \mathbb{R}^{1\times n-1}$. Then x is not in the column space of S_1 because $y^t x = 1 \neq 0$. So, S is invertible. Moreover, $y^t S = [1, 0, \ldots, 0]$ so that y is the first row of S^{-1}. Now, if $S^{-1}BS = C$, then $SC = BS$ has first column equal $r(B)e_1$. Thus, the first column of C is $r(B)e_1$. Similarly, the first column of $CS^{-1} = S^{-1}B$ equals $r(B)y^t$. Thus, the first row of C is $r(B)e_1^t$. Hence, $S^{-1}BS = [r(B)] \oplus B_1$ such that $r(B_1) < r(B)$. Assertion 2 follows.

Assertion 3 *The conclusion of Theorem 6.3.1 holds.*

Proof. Note that the vectors x and y in Assertion 3 are the left and right eigenvectors of A corresponding to a simple eigenvalue λ of A with $|\lambda| = r(A)$. Now, $Ax = \lambda x$ implies that $\lambda = r(A)$. So, $S^{-1}AS = [r(A)] \oplus A_1$ such that $r(A_1) < r(A)$. Finally,

$$\lim_{m\to\infty}[A/r(A)]^m = \lim_{m\to\infty} S([1] \oplus (A_1/r(A))^m)S^{-1} = S([1] \oplus 0_{n-1})S^{-1} = xy^t.$$

□

In general, for any nonnegative matrix $A \in M_n$, we can consider $A_\varepsilon = A + \varepsilon ee^t$ for some positive $\varepsilon > 0$ so that the resulting matrix is positive so that $r(A_\varepsilon)$ is a simple eigenvalue of $A_\varepsilon)$ with positive left and right eigenvectors x_ε and y_ε. By continuity, we have the following.

Corollary 6.3.3 *Let $A \in M_n$ be a nonnegative matrix. Then $r(A)$ is an eigenvalue of A with at least one pair of nonnegative left and right eigenvector.*

For a nonnegative matrix A, $r(A)$ is call the Perron eigenvalue of A, and the corresponding nonnegative left and right eigenvectors are called the Perron eigenvectors.

Example 6.3.4 *Note that A^k is not positive for any positive integer k in all the following.*

If $A = I_2$, then $r(A) = 1$ and all nonzero vectors are left and right eigenvectors.

If $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, then $r(A) = 1$ with right and left eigenvectors $x = (1,0)^t/2$ and $y = (0,1)^t$.

If $A = \begin{pmatrix} 1/2 & 1/2 \\ 0 & 1 \end{pmatrix}$, then $r(A) = 1$ with right and left eigenvectors $x = (1,1)^t/2$ and $y = (0,2)^t$.

If $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$*, then* $r(A) = 1$ *with right and left eigenvectors* $x = (1,1)^t/2$ $y = (1,1)^t$.

A row (column) stochastic matrix is a matrix with nonnegative entries such that all row (colum) sums equal one. It appear in the study of Markov Chain in probability, population models, Google page rank matrix, etc. If $A \in M_n$ is both row and column stochastic, then it is doubly stochastic.

Corollary 6.3.5 *Let* A *be a row stochastic matrix. Then* $r(A) = 1$*. If* A^k *is positive, then* $r(A)$ *is a simple eigenvalue with a unique positive left eigenvector* x *satisfying* $e^t x = 1$*, and a unique positive left eigenvector* y *such that* $A^k \to xy^t$ *as* $k \to \infty$.

6.4 Kronecker (tensor) products

Definition 6.4.1 *Let* $A = (a_{ij}) \in M_{m,n}$, $B = (b_{rs}) \in M_{p,q}$*. Then* $A \otimes B = (a_{ij}B) \in M_{mp,ns}$.

Theorem 6.4.2 *The following equations hold for scalar* a, b *and matrices* $A, B, C, D)$ *provided that the sizes of the matrices are compatible with the described operations.*

(a) $(aA + bB) \otimes C = aA \otimes C + bB \otimes C$, $C \otimes (aA + bB) = aC \otimes A + bC \otimes B$.

(b) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.

Proof. (a) By direct verification. (b) Suffices to show $(A \otimes B)(C_j \otimes D_k) = (AC_j) \otimes (BD_k)$ for all columns C_j of C and D_k of D. □

Corollary 6.4.3 *Let* A, B *be matrices. Then* $f(A \otimes B) = f(A) \otimes f(B)$ *for* $f(X) = \overline{X}, X^t$ *or* X^*.

(a) *If* A, B *are invertible, then* $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

(b) *If* A *and* B *are unitary, then so is* $A \otimes B$ *with inverse* $(A \otimes B)^* = A^* \otimes B^*$.

(c) *If* $S^{-1}AS$ *and* $T^{-1}BS$ *are in triangular forms, then so is* $(S \otimes T)^{-1}(A \otimes B)(S \otimes T)$.

(d) *If* A *has eigenvalues* $a_1, \ldots, a_m$ *and* B *has eigenvalues* $b_1, \ldots, b_n$*, then* $A \otimes B$ *has eigenvalues*

$a_i b_j$ *with* $1 \le i \le m, 1 \le j \le n$*; if* x_i, y_j *are eigenvectors such that* $Ax_i = a_i x_i$ *and* $By_j = b_j y_j$,

then $(A \otimes B)(x_i \otimes j) = a_i b_j (x_i \otimes y_j)$.

(e) *If* A *and* B *have singular value decomposition* $A = U_1 D_1 V_1^*$ *and* $B = U_2 D_2 V_2^*$*, then the equation*

$(A \otimes B)(V_1 \otimes V_2) = (U_1 \otimes U_2)(D_1 \otimes D_2)$ *will yield the information for singular values and*

singular vectors.

We have the following application of the tensor product results to matrix equations.

Theorem 6.4.4 *Let $A \in M_m, B \in M_n$ and $C \in M_{m,n}$. Then the matrix equation*

$$AX + XB = C \qquad X \in M_{m,n}$$

can be rewritten as $(I_m \otimes A)\text{vec}(X) + (B^t \otimes I_n)\text{vec}(X) = \text{vec}(C)$*, where for $Z \in M_{m,n}$ we have* $\text{vec}(Z) \in \mathbb{C}^{mn}$ *with the first column of Z as the first m entries, second column of Z as the next m entries, etc.*

Consequently, the matrix equation is solvable if and only if $\text{vec}(C)$ *lies in the column space of $I_n \otimes A + B^t \otimes I_m$. In particular, if $I_n \otimes A + B^t \otimes I_m$ is invertible, then the matrix equation is always solvable.*

The Hadamard (Schur) product of two matrices $A = (a_{ij}), B = (b_{ij}) \in M_{m,n}$ is defined by $A \circ B = (a_{ij}b_{ij})$.

Corollary 6.4.5 *Let $A, B \in M_{m,n}$.*

(a) *Then $s_k(A \otimes B) \geq s_k(A \circ B)$ for $k = 1, \ldots, m$.*

(b) *If $m = n$, then $s_{n-k+1}(A \circ B) \geq s_{n^2-k+1}(A \otimes B)$ for $k = 1, \ldots, n$.*

(c) *If A, B are positive semidefinite, then so is $A \circ B$.*

Remark Note that if $A, B \in M_n$ are invertible, unitary, or normal, it does not follow that $A \circ B$ has the same property.

6.5 Compound matrices

Let $A \in M_{m,n}$ and $k \le \min\{m, n\}$. Then the compound matrix $C_m(A)$ is of size $\binom{m}{k} \times \binom{n}{k}$ with rows labeled by increasing subseqeuence $r = (r_1, \ldots, r_k)$ of $(1, \ldots, m)$ and columns labeled by increasing subseqeuence $s = (s_1, \ldots, s_k)$ of $(1, \ldots, n)$ in lexicographic order such that the (r, s) entry of $C_m(A)$ equals $\det(A[r, s])$, where $A[r, s] \in M_k$ is the submatrix of A with rows and columns indexed r and s, arranged in lexicographic order.

Example 6.5.1 *Let $A \in M_4$. Then $C_2(A) \in M_6$ with $(r_1, r_2), (s_1, s_2)$ entry equal to* $\det(A[r_1, r_2; s_1, s_2])$.

It is easy to check that $C_k(A^t) = C_k(A)^t$, $C_k(A^*) = C_k(A)^*$, etc.

We will prove a product formula for the compound matrix. The proof depends on the following result which generalizes the Cauchy-Binet formula.

Theorem 6.5.2 *Let $A \in M_{m,n}$ and $B \in M_{n,m}$. Then for any $1 \le k \le m$, the sum of the $k \times k$ principal minors of AB is the same as that of $BA \in M_n$.*

Note that when $k = m \le n$, the above result is known as the Cauchy Binet formula.

Proof. Recall that if

$$P = \begin{pmatrix} AB & 0 \\ B & 0_n \end{pmatrix}, \quad Q = \begin{pmatrix} 0_m & 0 \\ B & BA \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix},$$

then S is invertible and

$$PS = \begin{pmatrix} AB & 0 \\ B & 0_n \end{pmatrix} \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} AB & ABA \\ B & BA \end{pmatrix} = \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} \begin{pmatrix} 0_m & 0 \\ B & BA \end{pmatrix} = SQ$$

Thus, P and Q are similar, and

$$z^m \det(zI_n - BA) = \det(zI_{m+n} - Q) = \det(zI_{m+n} - P) = z^n \det(zI_m - AB).$$

Thus the sum of the kth principal minors of P and that of Q are the same. Evidently, the sum of the kth principal minors of P are the same as that of AB, and the sum of the kth principal minors of Q are the same as that of BA. The result follows. $\square$

Theorem 6.5.3 *Let $A \in M_{m,n}$, $B \in M_{n,p}$ and $k \le \min\{m,n,p\}$. Then $C_k(AB) = C_k(A)C_k(B)$.*

Proof. Let $\Gamma_{r,k}$ be the set of length k increasing subsequence of $(1,\dots,r)$ for $r \ge k$. Consider the entry of $C_k(AB)$ with row indexes $r = (r_1,\dots,r_k) \in \Gamma_{m,k}$ and column indexes $s = (s_1,\dots,s_k) \in \Gamma_{n,k}$. Let $\hat{A} \in M_{k,n}$ be obtained from A by using its rows indexed by $(r_1,\dots,r_k)$, and let $\hat{B} \in M_{n,k}$ be obtained from B by using its columns indexed by $(s_1,\dots,s_k)$. Then the (r,s) entry of $C_k(AB)$ equals $\det(\hat{A}\hat{B}) = C_k(\hat{A})C_k(\hat{B})$ by the Cauchy Binet formula. Note that $C_k(\hat{A})C_k(\hat{B})$ is the (r,s) entry of $C_k(A)C_k(B)$. The result follows. □

Corollary 6.5.4 *Let $A \in M_n$ and $k \le n$.*

(a) *If A is invertible (unitary), then so is $C_k(A)$.*

(b) *Suppose $A = UTU^*$ is in triangular form. Then $C_k(A) = C_k(U)C_k(T)C_k(U^*)$, where $C_k(T)$ is in triangular form. Consequently, $C_k(A)$ has eigenvalues $\prod_{j=1}^k \lambda_{i_j}(A)$.*

(c) *Suppose $U^*AV = D$ with $D = \sum_{j=1}^n s_j(A)E_{jj}$, where U, V are unitary. Then*

$$C_k(U^*)C_k(A)C_k(V) = C_k(D).$$

Consequently, $C_k(A)$ has singular values $\prod_{j=1}^k s_{i_j}(A)$, $1 \le i_1 < \cdots < i_k \le n$.

Corollary 6.5.5 *Let $A \in M_n$ with eigenvalues $\lambda_1(A),\dots,\lambda_n(A)$ satisfying $|\lambda_1(A)| \ge \cdots \ge |\lambda_n(A)|$. Then*

$$\prod_{j=1}^k |\lambda_j(A)| \le \prod_{j=1}^k s_j(A) \qquad \text{for } j = 1,\dots,n.$$

6.6 Additive compound

Let $A \in M_n$ and $1 \le k \le n$, and

$$C_k(tI_n + A) = C_k(A) + tD_k(A) + t^2 D_{2,k} + t^{k-3} D_{3,k}(A) + \cdots + t^k I_{\binom{n}{k}}.$$

The matrix $D_k(A)$ is called the additive compound of A.

Note that $D_k(aA + bB) = aD_k(A) + bD_k(B)$ for any $a, b \in \mathbb{C}, A, B \in M_n$.

Theorem 6.6.1 *Let $A \in M_n$. Then $D_k(S^{-1}AS) = C_k(S)^{-1} D_k(A) C_k(S)$ so that A has eigenvalues $\sum_{j=1}^k \lambda_{i_j}(A)$ with $1 \le i_1 < \cdots < i_k \le n$. Consequently, if A is normal (Hermitian, positive semi-definite) then so is $D_k(A)$.*

Corollary 6.6.2 *Let $A \in M_n$ be Hermitian. Then*

$$\sum_{j=1}^k \lambda_{n-j+1}(A) \le \sum_{j=1}^k \lambda_j(A) \le \sum_{j=1}^k s_j(A).$$

Theorem 6.6.3 *Let $A, B \in M_n$. Then $D_k(AB) = D_k(AB - BA) = D_k(A)D_k(B) - D_k(B)D_k(A)$. Consequently, if A and B commute, then so do $D_k(A)$ and $D_k(B)$.*

Proof. The proof follows from the fact that $D_k(X)$ can be written as

$$V^* \left(\sum_{j=1}^k (\underbrace{I_n \otimes \cdots \otimes I_n}_{j-1} \otimes X \otimes \underbrace{I_n \otimes \cdots \otimes I_n}_{k-j} \right) V,$$

where $V \in M_{n^k \times \binom{n}{k}}$ such that $V^*V = I_{\binom{n}{k}}$ and the columns of V is a basis for the subspace of $\mathbb{C}^{n^k}$ spanned by

$$\left\{ \sum_{\sigma \in S_k} \chi(\sigma) e_{\sigma(i_1)} \otimes \cdots \otimes e_{\sigma(i_k)} : 1 \le i_1 < \cdots < i_k \le n \right\},$$

where $\chi(\sigma) = 1$ if $\sigma \in S_k$ is an even permutation and $\chi(\sigma) = -1$ otherwise. □

6.7 More block matrix techniques

Schur Complement Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ such that $A_{11} \in M_k$ is invertible. Then

$$\begin{pmatrix} I_k & 0 \\ -A_{21}A_{11}^{-1} & I_{n-k} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{pmatrix}.$$

The matrix $A_{22} - A_{21}A_{11}^{-1}A_{12}$ is the Schur complement of A with respect to Q_{11}. Clearly, it is useful for block Gaussian elimination. Also, if A is invertible, then the Schur complement if the $n-k$ by $n-k$ submatrix in A^{-1}.

If A^{-1} exists, then $A_{22} - A_{21}A_{11}^{-1}A_{12}$ is invertible and

$$A^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{pmatrix}^{-1} \begin{pmatrix} I_k & 0 \\ A_{21}A_{11}^{-1} & I_{n-k} \end{pmatrix} = \begin{pmatrix} \star & \star \\ \star & (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \end{pmatrix}.$$

So, $(A_{22} - A_{21}A_{11}^{1}A_{12})^{-1}$ is the $(n-k) \times (n-k)$ matrix in the right bottom block of A^{-1}.

Block Hermitian matrices Suppose $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ such that $A_{11} \in M_k$ is invertible.

If $S = \begin{pmatrix} I_k & 0 \\ -A_{12}A_{11}^{-1} & I_{n-k} \end{pmatrix}$, then $SAS^* = A_{11} \oplus (A_{22} - A_{21}A_{11}^{-1}A_{12})$.