Math 408 Advanced Linear Algebra Examination 1 Due: noon, 7 October.

1. Let $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_n$ be such that $A \in M_k$ is invertible.

   (a) Show that $\begin{pmatrix} I_k & 0 \\ -CA^{-1} & I_{n-k} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I_k & -A^{-1}B \\ 0 & I_{n-k} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix}$, and
   $$\det(T) = \det(A)\det(D - CA^{-1}B).$$

   (b) Show that T is invertible if and only if $D - CA^{-1}B$ is invertible, and
   $$T^{-1} = \begin{pmatrix} I_k & -A^{-1}B \\ 0 & I_{n-k} \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & (D - CA^{-1}B)^{-1} \end{pmatrix} \begin{pmatrix} I_k & 0 \\ -CA^{-1} & I_{n-k} \end{pmatrix}.$$

2. Let $A \in M_7$ with minimal polynomial $(x-1)^2(x-i)^3$. Determine all possible Jordan forms of A.

3. (a) Let $A = J_k(\lambda)$ with $k \geq 2$ and $\lambda = \mu^2 \neq 0$. Show that there is an invertible matrix $S \in M_k$ such that $S^{-1}AS = \lambda I_k + 2\mu N_k + N_k^2 = (\mu I_k + N_k)^2$, and hence $A = B^2$ for $B = S(\mu I_k + N_k)S^{-1}$.

   (b) Show that for every invertible matrix $T \in M_n$, there is $R \in M_n$ such that $T = R^2$.

4. Suppose $B = J_n(0)$. Then B^2 has Jordan form
   $$\begin{cases} J_k(0) \oplus J_k(0) & \text{if } n = 2k, \\ J_k(0) \oplus J_{k+1}(0) & \text{if } n = 2k+1. \end{cases}.$$
   Deduce that there is no matrix $T \in M_6$ such that $T^2 = J_2(0) \oplus J_4(0)$.

5. Show that $A, B \in M_2$ are unitarily similar if and only if $\operatorname{tr} A = \operatorname{tr} B, \operatorname{tr} A^2 = \operatorname{tr} B^2, \operatorname{tr} AA^* = \operatorname{tr} BB^*$.

   Hint: To prove the sufficiency, first show that A and B have the same eigenvalues, and hence by unitary similarities, the triangular forms T_1 and T_2 of A and B have the same diagonal entries. Then show that the $(1,2)$ entries of the two triangular matrices can be changed to the same nonnegative number by applying diagonal unitary similarities $D_1^* T_1 D_1$ and $D_2^* T_2 D_2$.

6. Let $w = e^{i2\pi/3}$, and $F = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w \end{pmatrix}$.

   (a) Show that F is unitary.

   (b) Suppose $P = E_{12} + E_{23} + E_{31} \in M_3$. Show that the $PF = FD$ with $D = \operatorname{diag}(1, w, w^2)$.

   (c) Suppose $A = a_1 I + a_2 P + a_3 P^2 = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_3 & a_1 & a_2 \\ a_2 & a_3 & a_1 \end{pmatrix}$. Determine F^*AF.

7. Suppose $B \in M_n$ is invertible. Show that $B = LV$ for some unitary V and lower triangular matrix L in M_n. Deduce that every invertible positive semidefinite matrix A can be written as $L_1 L_1^*$ such that L_1 is lower triangular.

8. (a) Show that $A \in M_n$ is normal if and only if $\|Av\| = \|A^*v\|$ for all $v \in \mathbb{C}^n$.

(b) Show that $A \in M_n$ is skew-symmetric if and only if $u^t A u = 0$ for all $u \in \mathbb{C}^n$.

Hint: For the sufficiency, write $A = S + K$ with $S = (A + A^t)/2$ is symmetric and $K = (A - A^t)/2$ is skew-symmetric. Show that $S = 0$.