1. Let $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_n$ be such that $A \in M_k$ is invertible.

   (a) Show that $\begin{pmatrix} I_k & 0 \\ -CA^{-1} & I_{n-k} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I_k & -A^{-1}B \\ 0 & I_{n-k} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix}$, and

   $$\det(T) = \det(A)\det(D - CA^{-1}B).$$

   (b) Show that T is invertible if and only if $D - CA^{-1}B$ is invertible, and

   $$T^{-1} = \begin{pmatrix} I_k & -A^{-1}B \\ 0 & I_{n-k} \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & (D - CA^{-1}B)^{-1} \end{pmatrix} \begin{pmatrix} I_k & 0 \\ -CA^{-1} & I_{n-k} \end{pmatrix}.$$

   Solution. Let $X = \begin{pmatrix} I_k & 0 \\ -CA^{-1} & I_{n-k} \end{pmatrix}$, $Y = \begin{pmatrix} I_k & -A^{-1}B \\ 0 & I_{n-k} \end{pmatrix}$, $Z = \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix}$.

   (a) Then $XTY = \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} I_k & -A^{-1}B \\ 0 & I_{n-k} \end{pmatrix} = Z$.

   Because the determinant of an (upper or lower) triangular matrix is the product of the diagonal entries. So, $\det(X) = \det(Y) = 1$.

   Also, $\det(Z)$= $\det \begin{pmatrix} A & 0 \\ 0 & I_{n-k} \end{pmatrix} \det \begin{pmatrix} I_k & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} = \det(A)\det(D - CA^{-1}B)$. Hence,

   $\det(T) = \det(X)\det(T)\det(Y) = \det(XTY) = \det(Z) = \det(A)\det(D - CA^{-1}B) = \det(T)$.

   (b) Note that T is invertible if and only if $0 \neq \det(T) = \det(A)\det(D - CA^{-1}B)$. It is given that A is invertible so that $\det(A) \neq 0$. So T is invertible if and only if $\det(D - CA^{-1}B) \neq 0$.

   Note that $\begin{pmatrix} A^{-1} & 0 \\ 0 & (D - CA^{-1}B)^{-1} \end{pmatrix} = Z^{-1} = (XTY)^{-1} = Y^{-1}T^{-1}X^{-1}$.

   Thus, $T^{-1} = Y \begin{pmatrix} A^{-1} & 0 \\ 0 & (D - CA^{-1}B)^{-1} \end{pmatrix} X$.

2. Let $A \in M_7$ with minimal polynomial $(x-1)^2(x-i)^3$. Determine all possible Jordan forms of A.

   Solution. Because the minimal polynomial of A is $(x-1)^2(x-i)^3$, the Jordan form has the form $J_2(1) \oplus J_3(i) \oplus X$, where X are the Jordan bocks of the remaining eigenvalues that may be (i) $1, 1$, (ii) $1, i$, (iii) i, i. So, $X = J_2(1), J_1(1) \oplus J_1(1), J_1(1) \oplus J_1(i), J_2(i), J_1(i) \oplus J_1(i)$.

3. (a) Let $A = J_k(\lambda)$ with $k \geq 2$ and $\lambda = \mu^2 \neq 0$. Show that there is an invertible matrix $S \in M_k$ such that $S^{-1}AS = \lambda I_k + 2\mu N_k + N_k^2 = (\mu I_k + N_k)^2$, and hence $A = B^2$ for $B = S(\mu I_k + N_k)S^{-1}$.

   (b) Show that for every invertible matrix $T \in M_n$, there is $R \in M_n$ such that $T = R^2$.

Solution. (a) Note that $(\mu I_k + N_k)^2 - \lambda I = 2\mu N_k + N_k^2$ has rank $k-1$. So, $(\mu I_k + N_k)^2$ has Jordan form $J_k(\lambda) = A$. Thus, there is an invertible S such that $A = SB^2S^{-1} = (SBS^{-1})^2$.

(b) Suppose T is invertible. Then $T = R(J_1 \oplus \cdots \oplus J_m)R^{-1}$ such that J_i is a Jordan block with nonzero eigenvalue (as T is invertible). Thus, $J_i = B_i^2$ by (a), and $T = R(B_1^2 \oplus \cdots \oplus B_m^2)R^{-1} = (R\tilde{B}R^{-1})^2$ with $\tilde{B} = B_1 \oplus \cdots \oplus B_m$.

4. Suppose $B = J_n(0)$. Then B^2 has Jordan form

$$\begin{cases} J_k(0) \oplus J_k(0) & \text{if } n = 2k, \\ J_k(0) \oplus J_{k+1}(0) & \text{if } n = 2k+1. \end{cases}.$$

Deduce that there is no matrix $T \in M_6$ such that $T^2 = J_2(0) \oplus J_4(0)$.

Solution. Suppose there is T such that $T^2 = J_2(0) \oplus J_4(0)$. Then all eigenvalues of T are zero, and T cannot have more than one Jordan blocks in its Jordan form. Else, T^2 will have more than 2 Jordan blocks in its Jordan form by (a). Thus, T has Jordan form $J_6(0)$. But then T^2 will have Jordan form $J_3(0) \oplus J_3(0)$.

5. Show that $A, B \in M_2$ are unitarily similar if and only if $\operatorname{tr} A = \operatorname{tr} B, \operatorname{tr} A^2 = \operatorname{tr} B^2, \operatorname{tr} AA^* = \operatorname{tr} BB^*$.

Solution. Suppose A and B are unitarily similar. By Specht's theorem, $\operatorname{tr} w(A, A^*) = \operatorname{tr} w(B, B^*)$ for any words $w(x, y)$.

Conversely, suppose $\operatorname{tr} A = \operatorname{tr} B, \operatorname{tr} A^2 = \operatorname{tr} B^2, \operatorname{tr} AA^* = \operatorname{tr} BB^*$. If A has eigenvalues a_1, a_2 and B has eigenvalues b_1, b_2, then the first two equalities imply $a_1 + a_2 = b_1 + b_2$ and $a_1^2 + a_2^2 = b_1^2 + b_2^2$. Thus, $(a_1 + a_2)^2 = (b_1 + b_2)^2$ and $a_1^2 + a_2^2 = b_1^2 + b_2^2$, and hence $a_1 a_2 = b_1 b_2$, i.e., $\det(A) = \det(B)$. So,

$$\det(zI - A) = z^2 - (\operatorname{tr} A)z + \det(A) = z^2 - (\operatorname{tr} B)z + \det(B) = \det(zI - B).$$

Thus, A and B have the same eigenvalues, and there are unitary $U, V \in M_2$ such that $UAU^* = \begin{pmatrix} a_1 & a \\ 0 & a_2 \end{pmatrix}$ and $VBV^* = \begin{pmatrix} a_1 & b \\ 0 & a_2 \end{pmatrix}$. Let $a = |a|e^{ir}$ and $b = |b|e^{is}$. Then for $D_1 = \operatorname{diag}(e^{ir}1), D_2 = \operatorname{diag}(e^{is}, 1)$, we have

$$\tilde{A} = D_1 UAU^* D_1^* = \begin{pmatrix} a_1 & |a| \\ 0 & a_2 \end{pmatrix} \text{ and } \tilde{B} = D_2 VBV^* D_2^* = \begin{pmatrix} a_1 & |b| \\ 0 & a_2 \end{pmatrix}.$$

Now, $|a_1|^2 + |a_2|^2 + |a|^2 = \operatorname{tr}(\tilde{A}\tilde{A}^*) = \operatorname{tr}(AA^*) = \operatorname{tr}(BB^*) = \operatorname{tr}(\tilde{B}\tilde{B}^*) = |a_1|^2 + |a_2|^2 + |b|^2$. Thus $|a| = |b|$, and $D_1 UAU^* D_1^* = \tilde{A} = \tilde{B} = D_2 VBV^* D_2^*$ so that $A = U^* D_1^* D_2 VBV^* D_2^* D_1 U$.

6. Let $w = e^{i2\pi/3}$, and $F = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w \end{pmatrix}$.

(a) Show that F is unitary.

(b) Suppose $P = E_{12} + E_{23} + E_{31} \in M_3$. Show that the $PF = FD$ with $D = \text{diag}\,(1, w, w^2)$.

(c) Suppose $A = a_1 I + a_2 P + a_3 P^2 = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_3 & a_1 & a_2 \\ a_2 & a_3 & a_1 \end{pmatrix}$. Determine F^*AF.

Solution. (a) Let v_1, v_2, v_3 be the three columns of F. Since $1 + w + w^2 = 0$, we see that $0 = \langle v_1, v_2 \rangle = \langle v_1, v_2 \rangle$. Also, $\langle v_2, v_3 \rangle = 1 + w^4 + w^2 = 1 + w + w^2 = 0$. Furthermore, $\langle v_j, v_j \rangle = (1 + 1 + 1)/3 = 1$. So, the columns of F form an orthonormal basis. Hence F is unitary.

(b) By the fact that $w^3 = 1$, $PF = \begin{pmatrix} 1 & w & w^2 \\ 1 & w^2 & w \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & w & w^2 \\ 1 & w^2 & w^4 \\ 1 & w^3 & w^3 \end{pmatrix} = FD.$

(c) Now, $F^*PF = D = \text{diag}\,(1, w, w^2)$ and $F^*P^2F = (F^*PF)(F^*PF) = D^2 = \text{diag}\,(1, w^2, w)$. So, $F^*(a_1 I + a_2 P + a_3 P^2)F = a_1 I + a_2 D + d_3 D^2 = \text{diag}\,(a_1 + a_2 + a_3, a_1 + a_2 w + a_3 w^2, a_1 + a_2 w^2 + a_3 w)$.

7. Suppose $B \in M_n$ is invertible. Show that $B = LV$ for some unitary V and lower triangular matrix L in M_n. Deduce that every invertible positive semidefinite matrix A can be written as $L_1 L_1^*$ such that L_1 is lower triangular.

   Solution. Applying the Gram Schmidt process to the columns of the invertible matrix B^t, we see that $B^t = WR$for some unitary W and upper triangular matrix R. Then $B = R^t W^t = LV$ so that $V = R^t$ is in lower triangular form and $V = W^t$ is unitary.

   Now, if A is positive definite, then $A = BB^*$. Because $0 \neq \det(A) = \det(B)\det(B^*)$ so that $\det(B) \neq 0$. Hence, B is invertible and $B = LV$. Hence, $A = BB^* = LVV^*L^* = LL^*$.

8. (a) Show that $A \in M_n$ is normal if and only if $\|Av\| = \|A^*v\|$ for all $v \in \mathbb{C}^n$.

   (b) Show that $A \in M_n$ is skew-symmetric if and only if $u^t A u = 0$ for all $u \in \mathbb{C}^n$.

   Solution. (a) If $A^*A = AA^*$, then $\|Av\|^2 = v^*A^*Av = v^*AA^*v = \|A^*v\|^2$ for all $v \in \mathbb{C}^n$.

   Conversely, if $0 = \|Av\|^2 - |A^*v\|^2 = v^*(A^*A - AA^*)v$ for all $v \in \mathbb{C}^n$, then the matrix $B = A^*A - AA^*$ is Hermitian. So, $B = B^* = (A^*A - AA^*)^* = AA^* - A^*A = -B$. Thus, $B = 0$.

   (b) If $A = -A^t$, then for any $v \in \mathbb{C}^n$, if $\mu = \mu^t = (v^t A v)^t = v^t A^t v = v^t(-A)v = -v^t A v = -\mu$. Thus, $\mu = 0$.

   Conversely, suppose $v^t A v = 0$ for all $v \in \mathbb{C}^n$. Let $A = S + K$ such that $S = (A + A^t)/2, K = (A - A^t)/2$ so that S is symmetric and K is skew symmetric. Then $v^t K v = 0$ for all $v \in \mathbb{C}^n$ by the previous part of the proof. So, $v^t S v = 0$ for all $v \in \mathbb{C}^n$. If S is nonzero, then the largest singular value s_1 of S is nonzero, there is a unit vector u such that $u^t S u = s_1 > 0$. This is a contradiction. So, $S = 0$ and $A = K$ is slew-symmetric.