Math 408 Advanced Linear Algebra **Final examination** **Due: Dec. 8, noon**

The scores of the best 7 questions will go to the final examination grade.
The rest will be extra homework credits. Good luck!

1. (a) Use Gershgorin Theorem to conclude that $\begin{pmatrix} 1 & 1/2 & 1 \\ i & 4 & 0 \\ 0 & i/2 & 7 \end{pmatrix}$ has 3 distinct eigenvalues.

   (b) For $B = (b_{ij})$, let $G_i(B) = \{\mu \in \mathbb{C} : |\mu - b_{ii}| \le \sum_{j \ne i} |b_{ij}|\}$. Construct an example of B so that there is no eigenvalue in $G_1(B)$.

2. Let $A \in M_{n,p}$ and $B \in M_{n,q}$. Show that

$$\text{rank}\,([A|B]) = \text{rank}\,(A) + \text{rank}\,(B) - d,$$

   where d is the dimension of $Col(A) \cap Col(B)$, the intersection of the column space of A and the column space of B.

   Hint: Let $\{v_1, ..., v_d\}$ be a basis for $Col(A) \cap Col(B)$. Form bases for $Col(A), Col(B)$ containing $\{v_1, \dots, v_d\}$, and ....

3. Let $A \in M_n$ be a normal matrix.

   (a) If $B \in M_m$ is normal, show that $A \otimes B$ is normal.

   (b) If $m > 1$ and $C \in M_m$ is not normal, show that $A \otimes C$ is not normal.

4. Prove or disprove the following.

   (a) A principal submatrix of a Hermitian matrix $A \in M_n$ is Hermitian.

   (b) A principal submatrix of a normal matrix is normal.

5. Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in M_n$ such that $A_{11} \in M_k$ is invertible.

   (a) Compute RA for $R = \begin{pmatrix} I_k & 0 \\ -A_{21}A_{11}^{-1} & I_{n-k} \end{pmatrix}$.

   (b) Show that A is invertible if and only if $C = A_{22} - A_{21}A_{11}^{-1}A_{12}$ is invertible, and A^{-1} has the form $A^{-1} = \begin{pmatrix} * & * \\ * & C^{-1} \end{pmatrix}$.

   Hint: If $T = RA = \begin{pmatrix} T_{11} & * \\ 0 & T_{22} \end{pmatrix}$ is invertible, then show that T^{-1} has the form $\begin{pmatrix} T_{11}^{-1} & * \\ 0 & T_{22}^{-1} \end{pmatrix}$.

6. Let $A \in M_n$ be Hermitian. Show that if $A \in M_n$ is positive semidefinite if and only if all principal minors of A are nonnegative.

7. (a) Show that if $A \in M_n$ is normal, then $r(A) = w(A) = s_1(A)$.

   (b) Construct a non-normal $B \in M_n$ such that $r(B) = w(B) = s_1(B)$.

8. Let $(a_1, b_1), \ldots, (a_n, b_n) \in \mathbb{C}^{1\times 2}$ be such that $a_1, \ldots, a_n$ are distinct.

   (a) Suppose $f(z) = f_{n-1}z^{n-1} + f_{n-2}z^{n-2} + \cdots + f_0$ satisfies $f(a_i) = b_i$ for $i = 1, \ldots, n$. Show that

   $$V(a_1, \ldots, a_n)(f_0, \ldots, f_{n-1})^t = (b_1, \ldots, b_n)^t,$$

   where the ith row of $V(a_1, \ldots, a_n) \in M_n$ is $(1\ a_i\ a_i^2\ \cdots\ a_i^{n-1})$ for $i = 1, \ldots, n$.

   (b) Show that $\det(V(a_1, \ldots, a_n)) = \prod_{1\le i<j\le n}(a_j - a_i)$; thus $V(a_1, \ldots, a_n)$ is always invertible.

   Hint: You may use induction, or show that $(a_j - a_i)$ is always a factor of $\det(V(a_1, \ldots, a_n))$ if we regard $\det(V(a_1, \ldots, a_n))$ as a multinomial of $(a_1, \ldots, a_n)$. Then conclude that the expansion of $\det(V(a_1, \ldots, a_n))$ is the same as the expansion of $\prod_{1\le i<j\le n}(a_j - a_i)$.

   Remark The matrix $V(a_1, \ldots, a_n)$ is known as the Vandetmonde matrix.

9. Let $A \in M_n$.

   (a) Use the fact that A contains all the eigenvalues of A and the convexity of the numerical range to show that $(\mathrm{t}r\ A)/n \in W(A)$.

   (b) Show that there is a unitary U such that U^*AU has all diagonal entries equal to $(\mathrm{t}r\ A)/n$.

   [Hint: Induction.]

10. Let $N_n = E_{12} + \cdots + E_{n-1,n} \in M_n$ and

    $$T_n = aI_n + bN_n + cN_n^t = \begin{pmatrix} a & b & & & \\ c & a & b & & \\ & \ddots & \ddots & \ddots & \\ & & c & a & b \\ & & & c & a \end{pmatrix}.$$

    (a) Show that $\det(T_1) = a, \det(T_2) = a^2 - bc$, $\det(T_n) = a\det(T_{n-1}) - bc\det(T_{n-2})$.

    (b) If $bc = 0$, show that $\det(T_n) = a^n$.

    (c) Suppose $bc \neq 0$ and α, β are the zeros of $f(z) = z^2 - az + bc$.

    (c.1) If $\alpha = \beta$, show that $\det(T_n) = (n + 1)(a/2)^n$.

    (c.2) If $\alpha \neq \beta$, show that $\det(T_n) = c_1\alpha^n + c_2\beta^n$ with $c_1 = \alpha/(\alpha - \beta)$ and $c_2 = -\beta/(\alpha - \beta)$.

    Hint: Suffices to show that the suggested values in (c.1), (c.2) satisfy the conditions in (a).