Math 408 Advanced Linear Algebra **Final examination** **Sample solution**

1. (a) Use Gershgorin Theorem to conclude that $\begin{pmatrix} 1 & 1/2 & 1 \\ i & 4 & 0 \\ 0 & i/2 & 7 \end{pmatrix}$ has 3 distinct eigenvalues.

   (b) For $B = (b_{ij})$, let $G_i(B) = \{\mu \in \mathbb{C} : |\mu - b_{ii}| \le \sum_{j \ne i} |b_{ij}|\}$. Construct an example of B so that there is no eigenvalue in $G_1(B)$.

   Solution. (a) The Girshgorin disks are $G_1(A) = \{\mu : |\mu - 1| \le 1.5\}$ $G_2(A) = \{\mu : |\mu - 4| \le 1\}$ and $G_3(A) = \{\mu : |\mu - 7| \le 1/2\}$. They are all disjoint so that each disk will contains an eigenvalue of A. Hence, A has 3 distinct eigenvalues.

   (b) Let $B = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$. Then $G_1(B) = \{\mu : |\mu| \le 1\}$ and $G_2(B) = \{\mu : |\mu| \le 4\}$. Note that B has eigenvalues $2, -2$. So, $G_1(B)$ does not contain any eigenvalues of B.

2. Let $A \in M_{n,p}$ and $B \in M_{n,q}$. Show that

   $$\mathrm{rank}\,([A|B]) = \mathrm{rank}\,(A) + \mathrm{rank}\,(B) - d,$$

   where d is the dimension of $Col(A) \cap Col(B)$, the intersection of the column space of A and the column space of B.

   Solution. Let $\{v_1, ..., v_d\}$ be a basis for $Col(A) \cap Col(B)$. Then we can find columns of A, say, $a_1, \dots, a_p$ such that $\{v_1, \dots, v_d, a_1, \dots, a_p\}$ form a basis for $Col(A)$, and we can find columns of B, say, $b_1, \dots, b_q$ such that $\{v_1, \dots, v_d, b_1, \dots, b_q\}$ for a basis for $Col(B)$. We claim that $\mathcal{B} = \{v_1, \dots, v_d, a_1, \dots, a_p, b_1, \dots, b_q\}$ form a basis for $Col([A|B])$. Clearly, every column of A lies in the span of $\{v_1, \dots, v_d, a_1, \dots, a_p\}$ and every column of B lies in the span of $\{v_1, \dots, v_d, b_1, \dots, b_p\}$. So $\mathcal{B}$ is a generating set of the column space of $[A|B]$. It remains to show that $\mathcal{B}$ is linearly independent. To this end suppose $0 = \sum_{j=1}^d \mu_j v_j + \sum_{j=1}^p \nu_j a_j + \sum_{j=1}^q \xi_j b_j$. Then $\sum_{j=1}^d \mu_j v_j + \sum_{j=1}^p \nu_j a_j = -\sum_{j=1}^q \xi_j b_j$ is a vector in $Col(A) \cup Col(B)$. Hence, $-\sum_{j=1}^q \xi_j b_j$ is a linear combination of $v_1, \dots, v_d$ so that the coefficients of $a_1, \dots, a_p$ must be zero. But then we have $0 = \sum_{j=1}^d \mu_j v_j + \sum_{j=1}^q \xi_j b_j$. Because $\{v_1, \dots, v_d, b_1, \dots, b_p\}$ is linearly independent, we see that $0 = \mu_1 = \cdots = \mu_d = \xi_1 = \cdots \xi_q$. Hence, $\mathcal{B}$ is a basis for $Col[A|B]$. Hence, $\mathrm{rank}\,[A|B] = d + p + q = (d + p) + (d + q) - d = \mathrm{rank}\,(A) + \mathrm{rank}\,(B) - d$.

   bases for $Col(A), Col(B)$ containing $\{v_1, \dots, v_d\}$, and ....

3. Let $A \in M_n$ be a normal matrix non equal to zero.

   (a) If $B \in M_m$ is normal, show that $A \otimes B$ is normal.

   (b) If $m > 1$ and $C \in M_m$ is not normal, show that $A \otimes C$ is not normal.

   Solution. (a) Suppose A, B are normal. Then $(A \otimes B)(A \otimes B)^* = (A \otimes B)(A^* \otimes B^*) = (AA^*) \otimes (BB^*) = (A^*A) \otimes (B^*B) = (A^* \otimes B^*)(A \otimes B) = (A \otimes B)^*(A \otimes B)$.

   (b) If C is not normal, then $(A \otimes C)(A \otimes C)^* = (A \otimes C)(A^* \otimes C^*) = (AA^*) \otimes (CC^*) = T \otimes CC^*$ and $(A \otimes C)^*(A \otimes C) = (A^* \otimes C^*)(A \otimes C) = (A^*A) \otimes (C^*C) = T \otimes C^*C$. As long as $T = AA^* = A^*A$ is nonzero, $CC^* \ne C^*C$ will imply that $(A \otimes C)(A \otimes C)^* \ne (A \otimes C)^*(A \otimes C)$.

4. Prove or disprove the following.

   (a) A principal submatrix of a Hermitian matrix $A \in M_n$ is Hermitian.

   (b) A principal submatrix of a normal matrix is normal.

   Solution. (a) If $A = (a_{ij})$ then $a_{ij} = \bar{a}_{ji}$ for all i, j. For principal submatrix $B = (b_{rs})$, if $b_{rs} = a_{ij}$ then $b_{sr} = \bar{a}_{ji}$. So, we have $b_{rs} = \bar{b}_{sr}$. Hence, $B = B^*$ is Hermitian.

   (b) Let $B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ Then B is normal as $BB^* = I = B^*B$. Clearly, the submatrix lying in rows and columns 1 and 2 is not normal.

5. Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in M_n$ such that $A_{11} \in M_k$ is invertible.

   (a) Compute RA for $R = \begin{pmatrix} I_k & 0 \\ -A_{21}A_{11}^{-1} & I_{n-k} \end{pmatrix}$.

   (b) Show that A is invertible if and only if $C = A_{22} - A_{21}A_{11}^{-1}A_{12}$ is invertible, and A^{-1}

   has the form $A^{-1} = \begin{pmatrix} * & * \\ * & C^{-1} \end{pmatrix}$.

   Solution. (a) Multiplying the matrices, we get $RA = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{pmatrix}$.

   (b) By the fact that $\det(A) = \det(A_{11})\det(C)$ and $\det(A_{11}) \neq 0$, we see that $\det(A) \neq 0$ if and only if $\det(C) \neq 0$.

   Because $A^{-1}R^{-1} = (RA)^{-1} = \begin{pmatrix} A_{11}^{-1} & * \\ 0 & C^{-1} \end{pmatrix}$, we have $A^{-1} = \begin{pmatrix} A_{11}^{-1} & X \\ 0 & C^{-1} \end{pmatrix} R = \begin{pmatrix} * & * \\ * & C^{-1} \end{pmatrix}$.

6. Let $A \in M_n$ be Hermitian. Show that if $A \in M_n$ is positive semidefinite if and only if all principal minors of A are nonnegative.

   Solution. Suppose A is positive semidefinite. Then A has nonnegative eigenvalues $a_1 \geq \cdots \geq a_n \geq 0$. For any principal submatrix B with eigenvalues $b_1, \ldots, b_k$, we know that $b_k \geq a_n \geq 0$. Thus, B has nonnegative eigenvalues and has nonegative determinant. Note that if A is positive definite, then all principal minors are positive.

   To prove the converse, we show by induction that if the leading $k \times k$ principal minors of a Hermitian matrix A is positive, then A is positive definite. If $n = 1$, the result is clear. Suppose the result holds for Hermitian matrices M_{n-1} for $n \geq 2$. Let $A \in M_n$ be Hermitian with positive principal minors. Remove the last row and last column of A to get B. Then $B \in M_{n-1}$ is Hermitian, and has positive principal minors. So, B is positive definite. Suppose A has eigenvalues $a_1 \geq \cdots \geq a_n$ and B has eigenvalues $b_1 \geq \cdots \geq b_{n-1} > 0$. By interlacing inequalities, $a_1 \geq b_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq b_{n-1} \geq a_n$. So, $a_{n-1} > 0$. But this follows readily from that fact that $\det(A) = a_1 \cdots a_n > 0$ and $a_1 \cdots a_{n-1} > 0$.

   Now, suppose $A \in M_n$ is Hermitian with nonnegative principal minors. If A has k nonzero eigenvalues, then $\det(zI - A) = z^{n-k}(z^k - c_1 z^{k-1} + \cdots + (-1)^k c_k)$, where c_k it the nonzero number equal to the product of the k. Thus, A has an invertible $k \times k$ submatrix A_1. If A_1 is not positive definite, then A_1 has a negative principal minor and so is A. Hence, A_1 is positive definite. Now, permutating the rows and columns of A, we may assume that $A = \begin{pmatrix} A_1 & B^* \\ B & C \end{pmatrix}$ and

$$RAR^* = \begin{pmatrix} A_1 & 0 \\ 0 & C - BA_1^{-1}B^* \end{pmatrix} = A_1 \oplus 0_{n-k} \quad \text{with } R = \begin{pmatrix} I_k & 0 \\ -BA_1^{-1} & I_{n-k} \end{pmatrix}$$

as A has rank k. Thus, $A = SS^*$ with $R_1 A_1^{1/2}$ is positive semidefinite, where R_1 is the $n \times k$ matrix consisting of the first k columns of R

7. (a) Show that if $A \in M_n$ is normal, then $r(A) = w(A) = s_1(A)$.

   (b) Construct a non-normal $B \in M_n$ such that $r(B) = w(B) = s_1(B)$.

   Solution. (a) Suppose A is normal. Then $A = UDU^*$ for some unitary U and $D = \text{diag}\,(a_1, \ldots, a_n)$. Assume $|a_1| \geq \cdots \geq |a_n|$. Then $AA^* = UDD^*U^*$ has eigenvalues $|a_1|^2 \geq \cdots \geq |a_n|^2$. Thus, $r(A) = |a_1| = s_1(A)$. It follows that $r(A) = w(A) = s_1(A)$.

   (b) Let $B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Clearly, $BB^* = \text{diag}\,(4, 1, 0)$ and $B^*B = \text{diag}\,(4, 0, 1)$ are different. So, B is not normal. Now, B has eigenvalues $2, 0, 0$, and singular values $2, 1, 0$. Thus, $r(B) = s_1(B)$.

8. Let $(a_1, b_1), \ldots, (a_n, b_n) \in \mathbb{C}^{1\times 2}$ be such that $a_1, \ldots, a_n$ are distinct.

   (a) Suppose $f(z) = f_{n-1}z^{n-1} + f_{n-2}z^{n-2} + \cdots + f_0$ satisfies $f(a_i) = b_i$ for $i = 1, \ldots, n$. Show that

   $$V(a_1, \ldots, a_n)(f_0, \ldots, f_{n-1})^t = (b_1, \ldots, b_n)^t,$$

   where the ith row of $V(a_1, \ldots, a_n) \in M_n$ is $(1\ a_i\ a_i^2\ \cdots\ a_i^{n-1})$ for $i = 1, \ldots, n$.

   (b) Show that $\det(V(a_1, \ldots, a_n)) = \prod_{1\leq i<j\leq n}(a_j - a_i)$; thus $V(a_1, \ldots, a_n)$ is always invertible.

   Solution. (a) Write the system of equation $f_0 + f_1 a_j + \cdots f_{n-1}a_j^{n-1} = b_j$ for $j = 1 \ldots, n$, in matrix form, we get the linear system.

   (b) If we subtract the $f(a_1, \ldots, a_n) = \det(V(a_1, \ldots, a_n))$ can be view as a multinomial of the variables $a_1, \ldots, a_n$. If we subtract the jth column from the ith column, we see that the ith column of the resulting matrix is a multiple of $(a_i - a_j)$. Hence, $\det(V(a_1, \ldots, a_n) = (a_i - a_j)g(a_1, \ldots, a_n)$ This is true for all $i < j$. Thus, $f(a_1, \ldots, a_n) = \prod_{1\ lei<j\leq n}(a_j - a_i)h(a_1, \ldots, a_n)$ for some multinomial of $h(a_1, \ldots, a_n)$. Now, comparing the coefficient of the $\prod_{j=1}^n a_j^{j-1}$, we see that $h(a_1, \ldots, a_n) = 1$.

   Remark The matrix $V(a_1, \ldots, a_n)$ is known as the Vandetmonde matrix.

9. Let $A \in M_n$.

   (a) Use the fact that A contains all the eigenvalues of A and the convexity of the numerical range to show that $(\text{tr}\,A)/n \in W(A)$.

   (b) Show that there is a unitary U such that U^*AU has all diagonal entries equal to $(\text{tr}\,A)/n$.

   Solution. (a) Note that $\lambda_1, \ldots, \lambda_n \in W(A)$. By the convexity of $W(A)$,

   $$(\text{tr}\,A)/n = (\lambda_1 + \cdots + \lambda_n)/n \in W(A).$$

(b) We prove the result by induction on n. The result is trivial if $n = 1$. Because $\gamma = (\mathrm{tr}\, A)/n \in W(A)$, there is a unit vector x such that $x^*Ax = (\mathrm{tr}\, A)/n$. Let X be unitary such that x is the first column of U. Then $X^*AX = \begin{pmatrix} \gamma & * \\ * & A_1 \end{pmatrix}$. Note that $A_1 \in M_{n-1}$ and $\mathrm{tr}\, A_1 = \mathrm{tr}\, A - \gamma = (n-1)\gamma$. By induction assumption, there is a unitary U_1 such that $U_1^* A_1 U_1$ has all diagonal entries equal to $(\mathrm{tr}\, A_1)/(n-1) = \gamma$. Let $U = X([1] \oplus U_1)$. Then U^*AU has diagonal entries $\gamma, \dots, \gamma$.

10. Let $N_n = E_{12} + \cdots + E_{n-1,n} \in M_n$ and

$$T_n = aI_n + bN_n + cN_n^t = \begin{pmatrix} a & b & & & \\ c & a & b & & \\ & \ddots & \ddots & \ddots & \\ & & c & a & b \\ & & & c & a \end{pmatrix}.$$

(a) Show that $\det(T_1) = a, \det(T_2) = a^2 - bc$, $\det(T_n) = a\det(T_{n-1}) - bc\det(T_{n-2})$.

(b) If $bc = 0$, show that $\det(T_n) = a^n$.

(c) Suppose $bc \neq 0$ and α, β are the zeros of $f(z) = z^2 - az + bc$.

(c.1) If $\alpha = \beta$, show that $\det(T_n) = (n+1)(a/2)^n$.

(c.2) If $\alpha \neq \beta$, show that $\det(T_n) = c_1\alpha^n + c_2\beta^n$ with $c_1 = \alpha/(\alpha - \beta)$ and $c_2 = -\beta/(\alpha - \beta)$.

Solution. (a) The result is clear for $n = 1, 2$. Expanding $\det(T_n)$ about the first row, we get the recurrence relation for $n \geq 3$

(b) If $bc = 0$, the matrix is in triangular form and the result follows.

(c) Note that $\alpha + \beta = a$ and $\alpha\beta = bc$. We prove the formula of $\det(T_n)$ by induction. One readily check that the formula of $\det(T_n)$ is valid if $n = 1, 2$.

(c.1) Suppose $\alpha = \beta = a/2$ so that $bc = (a/2)^2$. Then $n \geq 3$, by (a) we have

$$a\det(T_n) - bc\det(T_{n-1}) = a(n+1)(a/2)^n - bcn(a/2)^{n-1} = (n+2)(a/n)^{n+1} = \det(T_{n+1}).$$

(c.2) Suppose $\alpha \neq \beta$. Then $n \geq 3$, by (a) we have

$$a\det(T_n) - bc\det(T_{n-1}) = a(c_1\alpha^n + c_2\beta^n) + bc(c_1\alpha^{n-1} + c_2\beta^{n-1})$$

$$= (\alpha + \beta)(c_1\alpha^n + c_2\beta^n)/2 + \alpha\beta(c_1\alpha^{n-1} + c_2\beta^{n-1}) = c_1\alpha^{n+1} + c_2\beta^{n+1} = \det(T_{n+1}).$$