Math 408 Advanced Linear Algebra Homework 1 Your Name

1. (12 points) Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

   (a) Find the (complex) eigenvalues of A, and the corresponding eigenvectors.

   (b) Determine a complex matrix S such that $S^{-1}AS$ is in diagonal form.

   (c) Find a formula for A^k for $k = 1, 2, 3, \dots$.

2. (24 points) Suppose $A = SDS^{-1} \in M_n$ such that $D = \mathrm{diag}\,(\lambda_1, \dots, \lambda_n)$, and suppose S has columns $x_1, \dots, x_n$ and S^{-1} has rows $y_1^t, \dots, y_n^t$.

   (a) Show that $A^k = SD^kS^{-1}$.

   (b) Show that $A^k = \sum_{j=1}^n \lambda_j^k x_j y_j^t$ for every positive integer k.

   (c) If A is invertible, show that $A^k = SD^kS^{-1} = \sum_{j=1}^n \lambda_j^k x_j y_j^t$ for every negative integer k.

   (d) For any polynomial $f(z) = a_m z^m + \cdots + a_0$, let $f(A) = a_m A^m + \cdots + a_1 A + a_0 I_n$.
   Show that
   $$f(A) = Sf(D)S^{-1} = \sum_{j=1}^n f(\lambda_j) x_j y_j^t.$$

   (e) Show that $y_i^t x_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$

   (f) Suppose S has rows $u_1^t, \dots, u_n^t$ and S^{-1} has columns $v_1, \dots, v_n$. Show that $u_i^t v_j = \delta_{ij}$.

3. (12 points) Let $A \in M_n$ be diagonalizable with distinct eigenvalues $\lambda_1, \dots, \lambda_k$, and let
   $$f(z) = (z - \lambda_1) \cdots (z - \lambda_k).$$

   (a) Show that A is similar to $\lambda_1 I_{n_1} \oplus \cdots \oplus \lambda_k I_{n_k}$ with $n_1 + \cdots + n_k = n$.

   (b) Show that $f(A) = (A - \lambda_1 I) \cdots (A - \lambda_k I) = 0_n$.

   (c) If $g(z) = \det(zI - A) = (z - \lambda_1)^{n_1} \cdots (z - \lambda_k)^{n_k}$, show that $g(A) = 0_n$.

4. (12 points) Let $x(s) = (x_1(s), \dots, x_n(s))^t$ be such that $x_1(s), \dots, x_n(s)$ are differentiable functions. Suppose $x'(s) = (x_1'(s), \dots, x_n'(s))^t = Ax(s)$, where $A = SDS^{-1}$ with $D = \mathrm{diag}\,(\lambda_1, \dots, \lambda_n)$, and $y(s) = (y_1(s), \dots, y_n(s))^t = S^{-1}x(s)$

   (a) Show that $y'(s) = S^{-1}x'(s)$ so that $y'(s) = Dy(s)$.

   (b) Show that $y_i(s) = c_i e^{s\lambda_i}$ with $c_i = y_i(0)$ for $i = 1, \dots, n$.

   (c) Show that $x(s) = Sy(s) = S(c_1 e^{s\lambda_1}, \dots, c_n e^{s\lambda_n})^t$.

5. (Extra 6 points) Suppose $A \in M_m$ is upper triangular and $B \in M_n$ is lower triangular, and $C \in M_{m,n}$. Assume $X \in M_{m,n}$ satisfies $AX - XB = C$. Show that for $1 \leq j \leq n$,
   $$Col_j(C) = ACol_j(X) - \sum_{i=1}^n b_{ij} Col_i(X),$$
   where $Col_i(R)$ denotes the ith column of a matrix R.