Math 408 Advanced Linear Algebra Homework 2 Your Name

Six points for each questions

1. Let $A = \begin{pmatrix} 3 & 1 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

   (a) Determine S such that $S^{-1}AS$ is a direct sum of the Jordan block.

   (b) What is minimal polynomial of A?

   (c) Suppose $f(z)$ is a polynomial. What are the possible Jordan form of $f(A)$?

   Hint: Suppose $f(z) = m_A(z)q(z) + r(z)$. Then $r(z) = a_0 z + a_1$ because ....

   So, $f(A) = r(A)$ has Jordan form ....

2. If $A \in M_5$ has distinct eigenvalues $1, i$, determine all the possible Jordan forms of A.

   Hint: $\det(zI - A) = (x-1)^r(x-i)^s$ with $r, s > 0, r + s = 5$. So, ...

3. Suppose $A \in M_5$ is similar to $J_2(i) \oplus J_2(1) \oplus J_1(1)$. If $f(z)$ is a polynomial, what are the possible Jordan form of $f(A)$.

   Hint: Suppose $f(z) = m_A(z)q(z) + r(z)$. For each Jordan block $J_k(\lambda)$ determine $r(J_k(\lambda))$ depending on whether $r(\lambda) = 0$.

4. Suppose $f(z)$ is a polynomial, and $A \in M_n$.

   (a) If $Ax = \lambda x$ for a nonzero vector x, show that $f(A)x = f(\lambda)x$.

   (b) Show that an eigenvector of $f(A)$ may not be an eigenvector of A.

   Hint: (a) Show that $A^k = \lambda^k x$ for $k = 1, 2, \ldots$. Then consider general $f(z)$.

5. Suppose A is $m \times n$ and B is $n \times m$. Then AB and BA have the same set of nonzero eigenvalues of the same multiplicities.

   Hint: Show that $\begin{pmatrix} AB & 0 \\ B & 0_n \end{pmatrix} \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} \begin{pmatrix} 0_m & 0 \\ B & BA \end{pmatrix}$.

6. Suppose $f(z) = z^n + a_1 z^{n-1} + \cdots + a_n$. Then

$$A_f = \sum_{j=1}^{n-1} E_{j+1,j} - \sum_{j=1}^{n} a_j E_{1j} = \begin{pmatrix} -a_1 & -a_2 & & \cdots & -a_n \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & 0 \end{pmatrix}$$

   is the companion matrix of f. Here $\{E_{11}, E_{12}, \ldots, E_{nn}\}$ is the standard basis for M_n.

   (a) Show that $\det(zI - A_f) = f(z)$ by expanding $\det(zI - A_f)$ using the last row, and induction.

   (b) Show that $f(z)$ is the minimal polynomial of A_f.

   Hint: Show that $A - \lambda_i I$ has rank $n - 1$ for each distinct eigenvalue λ_i.

7. (Extra Credits) Suppose $A = J_m(\lambda)$ and $x'(s) = Ax(s)$. Show that the system of differential equation has a solution of the form:

$$y_k(s) = q_k(s)e^{s\lambda}, \quad k = 1, \ldots, m,$$

where $q_k(s) = c_{k0} + c_{k1}s + \cdots + c_{m-k,m-k}s^{m-k}$ is a polynomial in s of degree $m - k$.

Hint: The result is true for $k = m$. Then show that it is true for $k = m - 1, m - 2, ...$ by backward induction.