1. Let $A = \begin{pmatrix} 3 & 1 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

   (a) The only eigenvalue of A is 3 and the eigenvectors are the elements of

$$Nul(A - 3I) = Nul \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

   Note that $dim(Nul(A - 3I)) = 2$ and thus, there are two jordan blocks. The only possible 3×3 Jordan canonical form satisfying this is

$$J = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

   Next we need to find an invertible $S = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \in M_3$ satisfying $S^{-1}AS = J$, or equivalently $AS = SJ$. Thus $Av_1 = 3v_1$, $(A - 3)v_2 = v_1$ and $Av_3 = 3v_3$. We can take $v_1 = e_1$ and $v_3 = -2e_2 + e_3$. Note that $(A - 3)(-2e_1 + e_2) = e_1$. Thus, we can take $v_2 = -2e_1 + e_2$.

$$S = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

   (b) Based on the Jordan canonical form (JCF) obtained in (a), the minimal polynomial of A must be $m_A(x) = (x - 3)^2$.

   (c) From the Division/Euclidean algorithm, $f(z) = g(z)(z - 3)^2 + h(z)$, where $h(z)$ is a degree 1 polynomial, say $h(z) = az + b$ for some coefficients $a, b \in \mathbb{C}$. Then

$$f(A) = g(A) \underbrace{(A - 3I)^2}_{0_n} + h(A) = aA + bI = S(aJ - bI)S^{-1} = S \begin{pmatrix} 3a + b & a & 0 \\ 0 & 3a + b & 0 \\ 0 & 0 & 3a + b \end{pmatrix} S^{-1}$$

   If $a = 0$, then the Jordan canonical form of $f(A)$ is $\text{diag}(b, b, b)$. If $a \neq 0$,

$$f(A) = \underbrace{S \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{a} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{T} \underbrace{\begin{pmatrix} 3a + b & 1 & 0 \\ 0 & 3a + b & 0 \\ 0 & 0 & 3a + b \end{pmatrix}}_{JCF} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix} S^{-1}}_{T^{-1}}$$

2. We can determine the unique Jordan forms to permutation of the blocks. That is, there is no need to display all different permutations of the Jordan blocks.

   Case 1: If the characteristic polynomial of A is $\det(zI-A)=(z-1)^4(z-i)$.

$$\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&i\end{pmatrix},\begin{pmatrix}1&1&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&i\end{pmatrix},\begin{pmatrix}1&1&0&0&0\\0&1&0&0&0\\0&0&1&1&0\\0&0&0&1&0\\0&0&0&0&i\end{pmatrix},\begin{pmatrix}1&1&0&0&0\\0&1&1&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&i\end{pmatrix},\begin{pmatrix}1&1&0&0&0\\0&1&1&0&0\\0&0&1&1&0\\0&0&0&1&0\\0&0&0&0&i\end{pmatrix},$$

   Case 2: If the characteristic polynomial of A is $\det(zI-A)=(z-1)^3(z-i)^2$.

$$\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&i&0\\0&0&0&0&i\end{pmatrix},\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&i&1\\0&0&0&0&i\end{pmatrix},\begin{pmatrix}1&1&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&i&0\\0&0&0&0&i\end{pmatrix},$$

$$\begin{pmatrix}1&1&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&i&1\\0&0&0&0&i\end{pmatrix},\begin{pmatrix}1&1&0&0&0\\0&1&1&0&0\\0&0&1&0&0\\0&0&0&i&0\\0&0&0&0&i\end{pmatrix},\begin{pmatrix}1&1&0&0&0\\0&1&1&0&0\\0&0&1&1&0\\0&0&0&i&1\\0&0&0&0&i\end{pmatrix}$$

   Case 3: If the characteristic polynomial of A is $\det(zI-A)=(z-1)^2(z-i)^3$.

$$\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&i&0&0\\0&0&0&i&0\\0&0&0&0&i\end{pmatrix},\begin{pmatrix}1&1&0&0&0\\0&1&0&0&0\\0&0&i&0&0\\0&0&0&i&0\\0&0&0&0&i\end{pmatrix},\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&i&1&0\\0&0&0&i&0\\0&0&0&0&i\end{pmatrix},\begin{pmatrix}1&1&0&0&0\\0&1&0&0&0\\0&0&i&1&0\\0&0&0&i&0\\0&0&0&0&i\end{pmatrix},\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&i&1&0\\0&0&0&i&1\\0&0&0&0&i\end{pmatrix},\begin{pmatrix}1&1&0&0&0\\0&1&0&0&0\\0&0&i&1&0\\0&0&0&i&1\\0&0&0&0&i\end{pmatrix},$$

   Case 4: If the characteristic polynomial of A is $\det(zI-A)=(z-1)(z-i)^4$.

$$\begin{pmatrix}1&0&0&0&0\\0&i&0&0&0\\0&0&i&0&0\\0&0&0&i&0\\0&0&0&0&i\end{pmatrix},\begin{pmatrix}1&0&0&0&0\\0&i&1&0&0\\0&0&i&0&0\\0&0&0&i&0\\0&0&0&0&i\end{pmatrix},\begin{pmatrix}1&0&0&0&0\\0&i&1&0&0\\0&0&i&0&0\\0&0&0&i&1\\0&0&0&0&i\end{pmatrix},\begin{pmatrix}1&0&0&0&0\\0&i&1&0&0\\0&0&i&1&0\\0&0&0&i&0\\0&0&0&0&i\end{pmatrix},\begin{pmatrix}1&0&0&0&0\\0&i&1&0&0\\0&0&i&1&0\\0&0&0&i&1\\0&0&0&0&i\end{pmatrix},$$

3. Suppose $A=S(J_2(i)\oplus J_2(1)\oplus J_1(1))S^{-1}$. So, the Jordan form of $f(A)$ depends on $f(J_2(i)), f(J_2(1))$.

   For $f(J_2(i))=h(J_2(i))$ there are two possibilities depending on $f(z)=(z-i)^2q(z)+a_1z+b_0$.

   (1) If $a_1\neq 0$, then the Jordan form of $f(J_2(i))=J_2(f(i))$.

   (2) If $a_1=0$, then $f(J_2(i))=[f(i)]\oplus[f(i)]$.

   Similarly, for $f(J_2(1))$ there are also two possibilities depending on $f(z)=(z-1)^2p(z)+b_1z+b_0$.

   (1) If $b_1\neq 0$, then the Jordan form of $f(J_2(1))=J_2(f(1))$.

   (2) If $b_1=0$, then $f(J_2(1))=[f(1)]\oplus[f(1)]$.

   Consequently there are 4 possible forms for $f(A)=h(A)$ with eigenvalues $f(i), f(i), f(1), f(1), f(1)$,

   So, we have four possibilites for the JCF of $f(A)$.

   - If $(z-i)^2$ is not a factor of $f(z)$ and $(z-1)^2$ is not a factor of $f(z)$, then $f(A)$ is similar to
   $$J_2(f(i))\oplus J_2(f(1))\oplus J_1(f(1)).$$
   - If $(z-i)^2$ is not a factor of $f(z)$ and $(z-1)^2$ is a factor of $f(z)$, then $f(A)$ is similar to
   $$J_2(f(i))\oplus J_1(f(1))\oplus J_1(f(1))\oplus J_1(f(1)).$$

- If $(z-1)^2$ is a factor of $f(z)$ and $(z-1)^2$ is a not factor of $f(z)$, then A is similar to

$$J_1(f(i)) \oplus J_1(f(i)) \oplus J_2(f(1)) \oplus J_1(f(1)).$$

- If $(x-i)^2(x-1)^2$ is a factor of $f(z)$, then $f(A)=0$.

4. Suppose $f(z)$ is a polynomial, and $A \in M_n$.

   (a) First, note that for any positive integer k, we have

   $$A^k x = A^{k-1}(Ax) = A^{k-1}(\lambda x) = \lambda(A^{k-1}x) = \lambda(A^{k-2}(Ax)) = \lambda^2 A^{k-2}x = \cdots = \lambda^k x$$

   Suppose $f(z) = a_k z^k + \cdots + a_1 z + a_0$. Then

   $$\begin{aligned} f(A)x &= (a_k A^k + \cdots + a_1 A + a_0 I)x \\ &= a_k(A^k x) + \cdots + a_1(Ax) + a_0 x \\ &= a_k(\lambda^k x) + \cdots + a_1(\lambda x) + a_0 x \\ &= (a_k \lambda^k + \cdots + a_1 \lambda + a_0)x \\ &= f(\lambda)x \end{aligned}$$

   (b) Consider $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $f(z) = z^2$. Note that $x = e_2$ is an eigenvector of $f(A) = 0_2$ but $Ax = e_1 \neq \lambda x$.
   Alternatively, let $A = \operatorname{diag}(1,-1)$, then $A^2 = I$. Then every nonzero vector in $\mathbb{C}^2$ is an eigenvector of A^2, but it is not true for A.

5. Suppose A is $m \times n$ and B is $n \times m$. Without loss of generality, assume $m \geq n$. Define the $m+n$ matrices

   $$P = \begin{pmatrix} AB & 0 \\ B & 0_n \end{pmatrix}, \quad Q = \begin{pmatrix} 0_m & 0 \\ B & BA \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix}.$$

   Note that S is invertible since its upper triangular with nonzero diagonal entries. Furthermore

   $$PS = \begin{pmatrix} AB & 0 \\ B & 0_n \end{pmatrix} \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} AB & ABA \\ B & BA \end{pmatrix} = \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} \begin{pmatrix} 0_m & 0 \\ B & BA \end{pmatrix} = SQ$$

   Thus, P and Q are similar. Then

   $$z^m \det(zI_n - BA) = \det(zI_{m+n} - Q) = det(zI_{m+n} - P) = z^n \det(zI_m - AB)$$

   Thus $\det(zI_m - AB) = z^{m-n}\det(zI_n - BA)$. Thus, the nonzero eigenvalues of AB are exactly the nonzero eigenvalues of BA.

6. Suppose $f(z) = z^n + a_1 z^{n-1} + \cdots + a_n$. Let A_f below be the companion matrix of f.

   $$A_f = \begin{pmatrix} -a_1 & -a_2 & \cdots & \cdots & -a_n \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \implies zI - A_f = \begin{pmatrix} z+a_1 & a_2 & \cdots & \cdots & a_n \\ -1 & z & 0 & \cdots & 0 \\ 0 & -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & z \end{pmatrix}$$

(a) (Induction on the degree of f.) Suppose $f(z) = z + a_1$. Then $A_f = [-a_1]$ and hence $\det(zI - A_f) = z + a_1$. Let $n > 1$ an suppose that for any $g(z)$ of degree less than n, it holds that $\det(zI - A_g) = g$. Now let $f(z) = z^n + a_1 z^{n-1} + \cdots + a_n$, with . By cofactor expansion about the last column, we get

$$\begin{aligned}
\det(zI - A_f) &= z \det \begin{pmatrix} z + a_1 & a_2 & \cdots & \cdots & a_{n-1} \\ -1 & z & 0 & \cdots & 0 \\ 0 & -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & & \ddots & 0 \\ 0 & \cdots & 0 & -1 & z \end{pmatrix} + (-1)^{n+1} a_n \begin{pmatrix} -1 & z & 0 & \cdots & 0 \\ 0 & -1 & z & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & z \\ 0 & \cdots & 0 & 0 & -1 \end{pmatrix} \\
&= z \det(zI - A_g) + (-1)^{n+1} a_n (-1)^{n-1} = z \det(zI - A_g) + a_n,
\end{aligned}$$

where $g(z) = z^{n-1} + a_1 z^{n-2} + \cdots + a_{n-1}$. By the induction hypothesis $\det(zI - A_g) = g(z)$. Thus

$$\det(zI - A_f) = zg(z) + a_n = z(z^{n-1} + a_1 z^{n-2} + \cdots + a_{n-1}) + a_n = f(z).$$

(b) Suppose $\det(zI - A) = f(z) = (z - \lambda_1)^{n_1} \cdots (z - \lambda_k)^{n_k}$ for distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. Note that for each $i = 1, \ldots, k$, the number of Jordan blocks in the JCF of A corresponding to λ_i is given by $dim(Nul(A - \lambda_i)$.

$$A - \lambda_i I = \begin{pmatrix} a_1 - \lambda_i & -a_2 & \cdots & \cdots & -a_n \\ 1 & -\lambda_i & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & -\lambda_i \end{pmatrix}$$

The first $n-1$ columns of $A - \lambda_i I$ are clearly linearly independent. Hence rank $(A) \geq n-1$ and so $dim(Nul(A)) \leq 1$. Since λ_i is an eigenvalue, then $dim(Nul(A)) \geq 1$. Therefore $dim(Nul(A)) = 1$. This shows that for each $i = 1, \ldots, k$, the JCF of A only has one jordan block $J_{n_i}(\lambda_i)$. Thus $m_A(z) = (z - \lambda_1)^{n_1} \cdots (z - \lambda_k)^{n_k} = f(z)$.

7. (Extra Credits) Suppose $A = J_m(\lambda)$ and $x'(s) = Ax(s)$. That is

$$\begin{aligned}
x_1'(s) &= \lambda x_1(s) + x_2(s) \\
x_2'(s) &= \lambda x_2(s) + x_3(s) \\
\vdots \quad & \quad \vdots \qquad\quad \vdots \\
x_{m-1}'(s) &= \lambda x_{m-1}'(s) + x_m(s) \\
x_m'(s) &= \lambda x_m(s)
\end{aligned}$$

When $k = m$, the solution to the differential equantion $x_m'(s) = \lambda x_m(s)$ is

$$x_m(s) = q_m(s) e^{s\lambda}, \qquad \text{where } q_m(s) = c_m 0 = x_m(0) \text{ (constant function).}$$

Let $k < m$. Suppose that for $t = k+1, \ldots, m$, it holds that $x_t(s) = q_t(s) e^{s\lambda}$ for some degree $m - t$ polynomial $q_t(s)$. Now $x_k'(s) = \lambda x_k(s) + x_{k+1}(s)$, so

$$\begin{aligned}
x_k'(s) = \lambda x_k(s) + q_{k+1}(s) e^{s\lambda} \quad &\iff \quad e^{-s\lambda} x_k'(s) - \lambda e^{-s\lambda} x_k(s) = q_{k+1}(s) \\
&\iff \quad \tfrac{d}{ds}\left(e^{-s\lambda} x_k(s)\right) = q_{k+1}(s) \\
&\iff \quad e^{-s\lambda} x_k(s) = \textstyle\int q_{k+1}(s)\ ds \\
&\iff \quad x_k(s) = e^{s\lambda} \left(\textstyle\int q_{k+1}(s)\ ds\right)
\end{aligned}$$

Note that

$$\begin{aligned}\int q_{k+1}(s)\ ds &= \int (c_{(k+1)0} + c_{(k+1)1}s + \cdots + c_{m-k-1,m-k-1}s^{m-k-1})\ ds \\ &= c_{k0} + c_{(k+1)0}s + c_{(k+1)1}s^2 + \cdots + c_{m-k-1,m-k-1}s^{m-k} = q_k(s)\end{aligned}$$

Thus $x_k(s) = q_k(s)e^{s\lambda}$. By principle of mathematical induction, we obtain the desired conclusion.