Math 408 Advanced Linear Algebra Homework 3 Your Name

Six points for each questions

1. Let $A = \begin{pmatrix} 1 & 1-i & 2+i \\ 1 & 1+i & -2+i \\ i & i & 2 \end{pmatrix}$. Apply Gram Schmidt to the columns of A to get a unitary matrix U. Write $A = UR$ for an upper triangular matrix R.

2. Let $u = (1, 2i, 1-i)^t$. Find a unitary U with $u/\|u\|$ as the first column.

3. Recall that the trace of a matrix $A = (a_{ij}) \in M_n$ is defined by $\mathrm{tr}\, A = a_{11} + \cdots + a_{nn}$.

   (a) Show that if $X \in M_{m,n}$ and $Y \in M_{n,m}$, then $\mathrm{tr}\,(XY) = \mathrm{tr}\,(YX)$.

   (b) Show that if $B \in M_n$ has eigenvalues $\lambda_1, \ldots, \lambda_n$, then $\mathrm{tr}\, B = \sum_{j=1}^n \lambda_j$.

4. Let $A, B \in M_{m,n}$. Show that $\langle A, B\rangle = \mathrm{tr}\, AB^* = \sum_{i,j} a_{ij}\bar{b}_{ij}$ is a complex inner product satisfying

   (a) $\langle aA + bB, cC\rangle = a\bar{c}\langle A, C\rangle + b\bar{c}\langle B, C\rangle$ for any $A, B, C \in M_{m,n}$ and $a, b, c \in \mathbb{C}$.

   (b) $\langle A, B\rangle = \overline{\langle B, A\rangle}$.

5. Suppose $A = (a_{ij}) \in M_{m,n}$ has nonzero singular values $s_1, \ldots, s_k$. Show that $\sum_{j=1}^k s_j^2 = \mathrm{tr}\, AA^* = \sum_{i,j} |a_{ij}|^2$.

6. Suppose $A \in M_n$ has singular values $s_1 \geq \cdots \geq s_n$, and eigenvalues $\lambda_1, \ldots, \lambda_n$ with $|\lambda_1| \geq \cdots \geq |\lambda_n|$.

   (a) If A is normal, show that $(s_1, \ldots, s_n) = (|\lambda_1|, \ldots, |\lambda_n|)$.

   (b) If A is not normal, show that $\sum_{j=1}^n s_j^2 > \sum_{j=1}^n |\lambda_j|^2$.

   Hint: Suppose $U^*AU = (a_{ij})$ so that $a_{ii} = \lambda_i$ for $i = 1, \ldots, n$. Then $\sum_{j=1}^2 s_j^2 = \sum_{i,j} |a_{ij}|^2$ ...

7. Show that if $A, B \in M_n$ are unitarily similar, then A and B have the same eigenvalues and the same singular values.

8. Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} / \sqrt{2}$. Show that no two of the above matrices are unitarily similar.

9. (Optional) A word $W(X, Y)$ of $X, Y \in M_n$ is a product of the form $X^{n_1}Y^{n_2} \cdots X^{n_k}$ for some positive integers k and nonnegative integers $n_1, \ldots, n_k$ with $n_2, \ldots, n_{k-1} > 0$. Show that if A and B are unitarily similar, then $W(A, A^*)$ is unitarily similar to $W(B, B^*)$ so that $\mathrm{tr}\, W(A, A^*) = \mathrm{tr}\, W(B, B^*)$ for any word $W(X, Y)$.