Math 408 Advanced Linear Algebra Homework 3 Sample Solution

Six points for each questions

1. Let v_i be the i^{th} column of $A = \begin{pmatrix} 1 & 1-i & 2+i \\ 1 & 1+i & -2+i \\ i & i & 2 \end{pmatrix}$. Define

   - $\hat{u}_1 = v_1 = \begin{bmatrix} 1 \\ 1 \\ i \end{bmatrix} \implies \begin{array}{rcl} \|\hat{u}_1\| & = & \sqrt{3} \\ \langle v_2, \hat{u}_1 \rangle & = & (1-i) + (1+i) - i^2 = 3 \\ \langle v_3, \hat{u}_1 \rangle & = & (2+i) + (-2+i) - 2i = 0 \end{array}$

   - $\hat{u}_2 = v_2 - \frac{\langle v_2, \hat{u}_1 \rangle}{\|\hat{u}_1\|^2} \hat{u}_1 = \begin{bmatrix} 1-i \\ 1+i \\ i \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ i \end{bmatrix} = \begin{bmatrix} -i \\ i \\ 0 \end{bmatrix} \implies \begin{array}{rl} \|\hat{u}_2\| & = \sqrt{2} \\ \langle v_3, \hat{u}_2 \rangle & = i(2+i) - i(-2+i) = 4i \end{array}$

   - $\hat{u}_3 = v_3 - \frac{\langle v_3, \hat{u}_1}{\|\hat{u}_1\|^2} \hat{u}_1 - \frac{\langle v_3, \hat{u}_2}{\|\hat{u}_2\|^2} \hat{u}_2 = \begin{bmatrix} 2+i \\ -2+i \\ 2 \end{bmatrix} - \frac{0}{3} \hat{u}_1 - \frac{4i}{2} \begin{bmatrix} -i \\ i \\ 0 \end{bmatrix} = \begin{bmatrix} i \\ i \\ 2 \end{bmatrix} \implies \|\hat{u}_3\| = \sqrt{6}$

   Thus, we can take

   $$U = \begin{pmatrix} \frac{\hat{u}_1}{\|\hat{u}_1\|} & \frac{\hat{u}_2}{\|\hat{u}_2\|} & \frac{\hat{u}_3}{\|\hat{u}_3\|} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{i}{\sqrt{2}} & \frac{i}{\sqrt{6}} \\ \frac{i}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix}$$

   and

   $$R = U^* A = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-i}{\sqrt{3}} \\ \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 \\ \frac{-i}{\sqrt{6}} & \frac{-i}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 1 & 1-i & 2+i \\ 1 & 1+i & -2+i \\ i & i & 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{3}} & \frac{3}{\sqrt{3}} & 0 \\ 0 & \frac{2}{\sqrt{2}} & \frac{4i}{\sqrt{2}} \\ 0 & 0 & \frac{6}{\sqrt{6}} \end{pmatrix}$$

2. Let $u = (1, 2i, 1-i)^t$. We need to find $U = \begin{bmatrix} \frac{u}{\|u\|} & v & w \end{bmatrix}$ such that

   $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = UU^* = U^*U = \begin{bmatrix} 1 & \frac{1}{\|u\|}\langle v, u \rangle & \frac{1}{\|u\|}\langle w, u \rangle \\ \frac{1}{\|u\|}\langle u, v \rangle & \langle v, v \rangle & \langle w, v \rangle \\ \frac{1}{\|u\|}\langle u, w \rangle & \langle v, w \rangle & \langle w, w \rangle \end{bmatrix}.$$

   Note that v, w are must be unit vectors in $Span\{u\}^\perp$ and $v \perp w$.

   $$\begin{aligned} Span\{u\}^\perp &= \left\{ z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in \mathbb{C}^3 \;\middle|\; \langle z, u \rangle = z_1 - 2i(z_2) + z_3(1+i) = 0 \right\} \\ &= Span\left\{ x = \begin{bmatrix} -1+i \\ i \\ -1 \end{bmatrix}, y = \begin{bmatrix} -1+i \\ 1 \\ 1 \end{bmatrix} \right\} \end{aligned}$$

   Hence we can take $v = \frac{x}{\|x\|} = \frac{x}{2}$ and

   $$\hat{w} = y - \frac{\langle y, x \rangle}{\|x\|^2} x = \begin{bmatrix} -1+i \\ 1 \\ 1 \end{bmatrix} - \frac{1-i}{4} \begin{bmatrix} -1+i \\ i \\ -1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -4+2i \\ 3-i \\ 5-i \end{bmatrix}$$

and $w = \frac{\hat{w}}{||\hat{w}||} = \frac{4\hat{w}}{\sqrt{56}}$.

$$U = \begin{pmatrix} \frac{1}{\sqrt{7}} & \frac{-1+i}{2} & \frac{-4+2i}{\sqrt{56}} \\ \frac{2i}{\sqrt{7}} & \frac{i}{2} & \frac{3-i}{\sqrt{56}} \\ \frac{1-i}{\sqrt{7}} & \frac{-1}{2} & \frac{5-i}{\sqrt{56}} \end{pmatrix}.$$

3. Recall that the trace of a matrix $A = (a_{ij}) \in M_n$ is defined by $\text{tr}A = a_{11} + \cdots + a_{nn}$.

   (a) Let $X = [x_{rs}] \in M_{m,n}$ and $Y = [y_{pq}] \in M_{n,m}$, $XY = [w_{rq}] \in M_{m,m}$ and $YX = [z_{ps}] \in M_{n,n}$.

   $$\text{tr}(XY) = \sum_{r=1}^{m} w_{rr} = \sum_{r=1}^{m}\sum_{k=1}^{n} x_{rk}y_{kr} = \sum_{k=1}^{n}\sum_{r=1}^{m} y_{kr}x_{rk} = \sum_{k=1}^{n} z_{kk} = \text{tr}(YX)$$

   (b) Suppose $B \in M_n$ has eigenvalues $\lambda_1, \ldots, \lambda_n$. Then there B is similar to an upper triangular matrix T whose j^{th} diagonal entry is λ_j for $j = 1, \ldots, n$. That is $B = STS^{-1}$ for some nonsingular S. Then

   $$\text{tr}(B) = \text{tr}(\underbrace{S}_{X}\underbrace{TS^{-1}}_{Y}) = \text{tr}(\underbrace{TS^{-1}}_{Y}\underbrace{S}_{X}) = tr(T) = \lambda_1 + \cdots + \lambda_n$$

4. For $A = [a_{ij}], B = [b_{ij}] \in M_{m,n}$, define $\langle A, B\rangle = \text{tr}AB^* = \sum_{i,j} a_{ij}\bar{b}_{ij}$ and observe that

   $$\langle A, A\rangle = \sum_{i,j} a_{ij}\bar{a}_{ij} = \sum_{i,j} |a_{ij}|^2 \geq 0$$

   and $\langle A, A\rangle = 0$ if and only if $a_{ij} = 0$ for all i, j (that is $A = 0$).

   (a) Note that for any $A, B, C \in M_{m,n}$ and $a, b, c \in \mathbb{C}$,
   $$\begin{aligned} \langle aA + bB, cC\rangle &= \text{tr}((aA + bB)(cC)^*) = \text{tr}((aA + bB)(\bar{c}C^*)) \\ &= \text{tr}((aA)(\bar{c}C^*) + (bB)(\bar{c}C^*)) = \text{tr}((a\bar{c})(AC^*)) + \text{tr}((b\bar{c})(BC^*)) \\ &= (a\bar{c})\text{tr}(AC^*) + (b\bar{c})\text{tr}(BC^*) \\ &= a\bar{c}\langle A, C\rangle + b\bar{c}\langle B, C\rangle \end{aligned}$$

   (b) Note that complex conjugation has the following properties: $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ and $\overline{z_1 z_2} = \overline{z}_1\ \overline{z_2}$. Thus,

   $$\overline{\langle B, A\rangle} = \overline{\left(\sum_{i,j} b_{ij}\overline{a_{ij}}\right)} = \left(\sum_{i,j} \overline{b_{ij}}a_{ij}\right) = \left(\sum_{i,j} a_{ij}\overline{b_{ij}}\right) = \langle A, B\rangle$$

5. Suppose $A = (a_{ij}) \in M_{m,n}$ has nonzero singular values $s_1, \ldots, s_k$. Then there exists an $m \times m$ unitary matrix U, an $n \times n$ unitary matrix V such that

   $$A = U\begin{pmatrix} \text{diag}(s_1, \ldots, s_k) & 0_{k,n-k} \\ 0_{m-k,k} & 0_{m-k,n-k} \end{pmatrix}V^* \quad \Longrightarrow \quad AA^* = U\Big(\text{diag}(s_1, \ldots, s_k) \oplus 0_{m-k}\Big)U^*$$

   Thus, the eigenvalues of AA^* are $s_1^2, \ldots, s_k^2$, in addition to (m-k) zeroes. By problem 3b,

   $$\sum_{j=1}^{k} s_j^2 = \text{tr}AA^* = \sum_{i,j} |a_{ij}|^2.$$

6. Suppose $A = [a_{ij}] \in M_n$ has singular values $s_1 \geq \cdots \geq s_n$, and eigenvalues $\lambda_1, \ldots, \lambda_n$ with $|\lambda_1| \geq \cdots \geq |\lambda_n|$.

   (a) If A is normal, then there exists a unitary U such that $A = U\text{diag}(\lambda_1, \ldots, \lambda_n)U^*$. The singular values of A are the square roots of the eigenvalues of

   $$AA^* = (U\text{diag}(\lambda_1, \ldots, \lambda_n)U^*)(U\text{diag}(\overline{\lambda_1}, \ldots, \overline{\lambda_n})U^*) = U\text{diag}(|\lambda_1|^2, \ldots, |\lambda_n|^2)U^*$$

   Thus $s_1^2 = |\lambda_1|^2, \ldots, s_n^2 = |\lambda_n|^2$. Then $s_j = |\lambda_j|$ for $j = 1, \ldots, n$.

   (b) Suppose A is not normal. Then there exists a unitary U such that $A = UTU^*$ where $T = [t_{ij}]$ is an upper triagular matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$. Furthermore, since A is not normal, then there should be $1 \leq i < j \leq n$ such that $t_{ij} \neq 0$. Recall that the eigenvalues of A are $s_1^2, \ldots, s_n^2$. Thus, by problem 3, we have

   $$\sum_j s_j^2 = \text{tr}(AA^*) = \text{tr}(U^*AA^*U) = tr(TT^*)$$

   From the definition of $\langle T, T\rangle = \text{tr}(TT^*)$ in problem 4, we also have

   $$\sum_j s_j^2 = \text{tr}(TT^*) = \sum_{i,j} |t_{ij}|^2 = \sum_j |\lambda_j|^2 + \sum_{i<j} |t_{ij}|^2 > \sum_j |\lambda_j|^2$$

7. Suppose A and B are unitarily similar. That is $A = UBU^*$ for some unitary U. Note that $AA^* = (UBU^*)(UBU^*) = U(BB^*)U^*$. Thus, AA^* and BB^* are unitarily similar. From previous lessons, we know that similar matrices have the same eigenvalues. Thus A and B have the same set of eigenvalues. Similarly, AA^* and BB^* have the same set of eigenvalues. Since the singular values of A (respectively, B) are the square roots of the eigenvalues of AA^* (respectively, BB^*), it follows that A and B have the same singular values.

8. Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} / \sqrt{2} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-3}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{pmatrix}$.

   - Consider the length-1 word $w(X, Y) = X$. Then

     $$\text{tr}(w(A, A^*)) = tr(A) = 4 \qquad \text{tr}(w(B, B^*)) = tr(B) = 4 \qquad \text{tr}(w(C, C^*)) = \text{tr}(C) = \frac{4}{\sqrt{2}}$$

     Thus C is cannot be unitarily similar to A nor B.

   - Consider the length-2 word $w(X, Y) = XY$. Then

     $$\text{tr}(w(A, A^*)) = tr(AA^*) = \text{tr} \begin{pmatrix} 5 & 6 \\ 6 & 9 \end{pmatrix} = 14 \neq \text{tr}(w(B, B^*)) = \text{tr}(BB^*) = \text{tr} \begin{pmatrix} 1 & 6 \\ 6 & 9 \end{pmatrix} = 10$$

     Thus, A is not unitarily similar to B.

9. (Optional) Suppose A and B are unitarily similar. Then there exists a unitary U such that $A = UBU^*$. We will show that for any word $W(X, Y)$ of $X, Y \in M_n$, we get

   $$W(A, A^*) = UW(B, B^*)U^*$$

   using induction on the length n of $W(X, Y)$.

   Suppose $n = 1$. We only have two cases: $W(X, Y) = X$ or $W(X, Y) = Y$.

- If $W(X,Y) = X$, then $W(A,A^*) = A = UBU^* = UW(B,B^*)U^*$.
- If $W(X,Y) = Y$, then $W(A,A^*) = A^* = (UBU^*)^* = UB^*U^* = UW(B,B^*)U^*$.

Now, assume that $\tilde{W}(A,A^*) = U\tilde{W}(B,B^*)U^*$ for any word $\tilde{W}$ of length $n < k$. Let $W(X,Y)$ be a word of length k. Then we consider two cases:

- If $W(X,Y) = X\tilde{W}(X,Y)$ for some length $k-1$ word $\tilde{W}(X,Y)$. Then by induction hypothesis

  $$W(A,A^*) = A\tilde{W}(A,A^*) = (UBU^*)(U\tilde{W}(B,B^*)U^*) = UB\tilde{W}(B,B^*)U^* = UW(B,B^*)U^*$$

- If $W(X,Y) = Y\tilde{W}(X,Y)$ for some length $k-1$ word $\tilde{W}(X,Y)$, then

  $$W(A,A^*) = A^*\tilde{W}(A,A^*) = (UB^*U^*)(U\tilde{W}(B,B^*)U^*) = UB^*\tilde{W}(B,B^*)U^* = UW(B,B^*)U^*$$

In both cases $W(A,A^*)$ is unitarily similar to $W(B,B^*)$. Therefore, $\mathrm{tr}(W(A,A^*)) = \mathrm{tr}(W(A,A^*))$.