Math 408 Advanced Linear Algebra Homework 4 Your Name

1. (12 points) Let $A = \begin{pmatrix} 1 & 1-i & i \\ 2 & 0 & 0 \end{pmatrix}$.

   (a) Find the nonzero singular values $s_1 \geq s_2 > 0$ of A.

   (b) Find the singular value decomposition of A,

   i.e., find unit vectors $u_1, u_2 \in \mathbb{C}^2$ and $v_1, v_2 \in \mathbb{C}^2$ such that $A = s_1 u_1 v_1^* + s_2 u_2 v_2^*$.

   (c) Find a positive definite $P \in M_2$ and $V \in M_{2,3}$ with $VV^* = I_2$ such that $A = PV$.

   (d) Find a positive definite $Q \in M_3$ and $U \in M_{2,3}$ with $UU^* = I_2$ such that $A = UQ$.

   (e) Find a unitary $U \in M_2, V \in M_3$ so that $U^*AV = s_1 E_{11} + s_2 E_{22} \in M_{2,3}$.

   Hint: For (a) and (b), find the eigenvalues and eigenvectors of AA^*.

2. (4 points) Show that a matrix $A \in M_n$ is positive semidefinite if and only if its singular values are the same as its eigenvalues.

3. (4 points) Let $A \in M_{m,n}$. Show that the number of nonzero singular values of A equals:

$$\operatorname{rank}(A) = \operatorname{rank}(AA^*) = \operatorname{rank}(A^*A).$$

4. (4 points) Let $A, B \in M_{m,n}$. Show that there exist unitary $U \in M_m$ and $V \in M_n$ such that $A = UBV$ if and only if A and B have the same singular values.

5. (8 points) Let $A = A^t \in M_n$ be a complex symmetric matrix. Suppose $u \in \mathbb{C}^n$ is a unit vector such that $|u^t Au|$ is maximum among all unit vectors.

   (a) Show that there is a unit vector v such that $v^t Av = \ |u^t Au|$.

   (b) Suppose U is unitary with v in part (a) as the first column. Show that $U^t AU = [v^t Av] \oplus A_1$

   such that $A_1 = A_1^t \in M_{n-1}$.

   (c) Show by induction that is a unitary $W \in M_n$ such that $W^t AW = \operatorname{diag}(s_1, \ldots, s_n)$.

   (d) Show that $s_1, \ldots, s_n$ in (c) are the singular values of A.

   Hint for (b): Suppose $a_{1j} = |a_{1j}|e^{ir} \neq 0$ for some $j > 1$. Let $x_\theta = \cos\theta e_1 + \sin\theta e_j e^{-ir}$. Show that $f'(0) > 0$ if

$$f(\theta) = |x_\theta^t (U^t AU) x_\theta|^2 = (a\cos^2\theta + 2|a_{1j}|\sin\theta\cos\theta + c_1\sin^2\theta)^2 + (c_2\sin^2\theta)^2,$$

   where $a_{jj}e^{-i2r} = c_1 + ic_2$ with $c_1, c_2 \in \mathbb{R}$. Then deduce that for sufficiently small $\theta > 0$, $f(\theta) > f(0) = |v^t Av|^2$ so that $|v_\theta^t Av_\theta| > |v^t Av|$ for the unit vector $v_\theta = Ux_\theta$.

6. (6 points) Let $A \in M_n$ be positive semidefinite with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$.

   (a) Show that $\lambda_1 = \max\{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}$.

   (b) Show that $\lambda_n = \min\{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}$.

   Hint: Suppose $A = UDU^*$ with $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)U^*$. Then $x^*Ax = y^*Dy = \sum_{j=1}^n \lambda_j |y_j|^2$ if $y = U^*x = (y_1, \ldots, y_n)^t$. Argue that $\sum_{j=1}^n |y_j|^2 = 1$ so that the maximum will occur when $|y_1|^2 = 1$ and minimum will occur when $|y_n|^2 = 1$.

7. (6 points) Let $A \in M_{m,n}$ with nonzero singular values $s_1 \geq \cdots \geq s_k > 0$. Use the result in the last question or otherwise to show that

$$s_1 = \max\{\|Av\| : v \in \mathbb{C}^n, \|v\| = 1\} \qquad \text{and} \qquad s_1 = \max\{\|A^*u\| : u \in \mathbb{C}^m, \|u\| = 1\}.$$

What can you can about the minimum values if $m, n > 1$?

Hint: Recall that $\|w\| = (w^*w)^{1/2}$ for a complex vector w so that $\|Av\| = (v^*A^*Av)^{1/2}$.

8. (Extra credit, 4 points) Suppose $A = -A^t$ is skew-symmetric. Suppose $u_1, u_2 \in \mathbb{C}^n$ are orthonormal pairs such that $|u_1^t A u_2|$ is maximum, and U is unitary with u_1, u_2 as the first two columns. Show that $U^tAU = \begin{pmatrix} 0 & a_{12} \\ -a_{12} & 0 \end{pmatrix} \oplus A_1$ for some $A_1 = -A_1^t \in M_{n-2}$.

Hint: Let $U^tAU = (a_{ij})$. If $a_{1j} \neq 0$, replace u_2 by $v_\theta = U(\cos\theta \mu_2 e_2 + \sin\theta \mu_j e_j)$, where $a_{12}\mu_2 = |a_{12}|$ and $a_{1j}\mu_j = |a_{1j}|$. Showo that $|u_1^t A v_\theta| > |u_1^t A u_2|$ for sufficiently small $\theta > 0$. If $a_{2j} \neq 0$, one can replace u_1 by $w_\theta = U(\cos\theta \xi_1 e_1 + \sin\theta \xi_2 e_2)$ and show that $|u_1 t^t A u_2|$ can be increased.