1. (12 points) Let $A = \begin{pmatrix} 1 & 1-i & i \\ 2 & 0 & 0 \end{pmatrix}$.

   (a) First compute

$$\det(AA^* - zI) = \det \begin{pmatrix} 4-z & 2 \\ 2 & 4-z \end{pmatrix} = (4-z)^2 - 4 = z^2 - 8z + 12 = (z-6)(z-2).$$

   Thus $s_1 = \sqrt{6}$ and $s_2 = \sqrt{2}$.

   (b) We want to find orthonormal vectors $u_1, u_2 \in \mathbb{C}^2$ and orthonormal vectors $v_1, v_2 \in \mathbb{C}^3$ such that $A = \sqrt{6}u_1v_1^* + \sqrt{2}u_2v_2^*$. The vectors v_1, v_2 are unit eigenvectors of A^*A corresponding to the eigenvectors 6 and 2 of AA^*, respectively. (Note eigenvectors corresponding to different eigenvalues are always orthogonal.)

$$Nul(A^*A - 6I) = Nul \begin{pmatrix} -1 & 1-i & i \\ 1+i & -4 & -1+i \\ -i & -1-i & -5 \end{pmatrix} = Nul \begin{pmatrix} 1 & 0 & -3i \\ 0 & 1 & 1-i \\ 0 & 0 & 0 \end{pmatrix} = Span \left\{ \begin{bmatrix} 3i \\ -1+i \\ 1 \end{bmatrix} \right\}$$

$$Nul(A^*A - 2I) = Nul \begin{pmatrix} 3 & 1-i & i \\ 1+i & 0 & -1+i \\ -i & -1-i & -1 \end{pmatrix} = Nul \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 1-i \\ 0 & 0 & 0 \end{pmatrix} = Span \left\{ \begin{bmatrix} -i \\ -1+i \\ 1 \end{bmatrix} \right\}$$

   So, we can let

$$v_1 = \frac{1}{2\sqrt{3}} \begin{bmatrix} 3i \\ -1+i \\ 1 \end{bmatrix} \quad \text{and} \quad v_2 = \frac{1}{2} \begin{bmatrix} -i \\ -1+i \\ 1 \end{bmatrix}.$$

   Then

$$u_1 = \frac{1}{\sqrt{6}} Av_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} i \\ i \end{bmatrix} \quad \text{and} \quad u_2 = \frac{1}{\sqrt{2}} Av_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} i \\ -i \end{bmatrix}.$$

$$A = \sqrt{6}u_1v_1^* + \sqrt{2}u_2v_2^* = \frac{1}{2} \begin{pmatrix} 3 & 1-i & i \\ 3 & 1-i & i \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -1 & 1-i & i \\ 1 & -1+i & -i \end{pmatrix}$$

   (c) From (b), we can write $A = YSW^*$ where

$$Y = \frac{1}{\sqrt{2}} \begin{pmatrix} i & i \\ i & -i \end{pmatrix}, \quad S = \begin{pmatrix} \sqrt{6} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} \frac{3i}{\sqrt{12}} & \frac{-i}{2} \\ \frac{-1+i}{\sqrt{12}} & \frac{-1+i}{2} \\ \frac{1}{\sqrt{12}} & \frac{1}{2} \end{pmatrix}.$$

   Since Y is unitary, we have $YY^* = I_2 = Y^*Y$ and $W^*W = I_2$. Then

$$A = YSW^* = YS(Y^*Y)W^* = \underbrace{(YSY^*)}_{P}\underbrace{(YW^*)}_{V}$$

   Then we can let $P = YSY^*$ and $V = YW^*$. Theorem 2.2.5 guarantees P is positive semidefinite and $VV^* = (YW^*)(WY^*) = Y(W^*W)Y^* = YY^* = I_2$. More explicitly,

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{3}+1 & \sqrt{3}-1 \\ \sqrt{3}-1 & \sqrt{3}+1 \end{pmatrix} \text{ and } V = \frac{i}{2\sqrt{6}} \begin{pmatrix} (-3+\sqrt{3})i & -(1+i)(1+\sqrt{3}) & 1+\sqrt{3} \\ -(3+\sqrt{3})i & -(1+i)(1-\sqrt{3}) & 1-\sqrt{3} \end{pmatrix}$$

(d) Similar to (c), we let $U = YW^*$ and $Q = WSW^*$. That is,

$$U = \frac{i}{2\sqrt{6}} \begin{pmatrix} (-3+\sqrt{3})i & -(1+i)(1+\sqrt{3}) & 1+\sqrt{3} \\ -(3+\sqrt{3})i & -(1+i)(1-\sqrt{3}) & 1-\sqrt{3} \end{pmatrix}$$

and

$$Q = \frac{1}{2\sqrt{6}} \begin{pmatrix} 3+\sqrt{3} & (1-i)(3-\sqrt{3}) & (3-\sqrt{3})i \\ (1+i)(3-\sqrt{3}) & 2(1+\sqrt{3}) & -(1-i)(1+\sqrt{3}) \\ -(3-\sqrt{3})i & -(1+i)(1+\sqrt{3}) & 1+\sqrt{3} \end{pmatrix}$$

So that

$$A = YSW^* = Y(W^*W)SW^* = (YW^*)(WSW^*) = UQ.$$

Note that $UU^* = (YW^*)(WY^*) = Y(W^*W)Y^* = YY^* = I_2$. Q is positive semidefinite since we can write it as $Q = BB^*$, where $B = W\sqrt{S}$ and $\sqrt{S}$ is just the diagonal matrix whose diagonal entries are square roots of the diagonal entries of S.

(e) We choose $U = Y$ and $V = \begin{bmatrix} W & w \end{bmatrix}$ where w is a unit vector orthogonal to the columns of W. The vector w is also an eigenvector of A^*A corresponding to 0.

$$Nul(A^*A) = Nul \begin{pmatrix} 5 & 1-i & i \\ 1+i & 2 & -1+i \\ -i & -1-i & 1 \end{pmatrix} = Nul \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2}+\frac{1}{2}i \\ 0 & 0 & 0 \end{pmatrix} = Span \left\{ \begin{bmatrix} 0 \\ 1-i \\ 2 \end{bmatrix} \right\}$$

Hence, we can let $w = \frac{1}{\sqrt{6}} \begin{bmatrix} 0 \\ 1-i \\ 2 \end{bmatrix}$. So,

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} i & i \\ i & -i \end{pmatrix}, \quad \text{and} \quad V = \begin{pmatrix} \frac{3i}{\sqrt{12}} & \frac{-i}{2} & 0 \\ \frac{-1+i}{\sqrt{12}} & \frac{-1+i}{2} & \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}}i \\ \frac{1}{\sqrt{12}} & \frac{1}{2} & \frac{2}{\sqrt{6}} \end{pmatrix}.$$

2. (4 points) Suppose $A \in M_n$ is positive semidefinite. Suppose A is positive semidefinite. Then there is unitary U such that $A = U\text{diag}\,(\lambda_1, \dots, \lambda_n)U$. Hence A^*A has eigenvalues $\lambda_1^2, \dots, \lambda_n^2$, and therefore has singular values $\lambda_1, \dots, \lambda_n$).

   To prove the converse, suppose $\lambda_j = s_j \geq 0$ for $j = 1, \dots, n$. From problem 6b of HW3, this implies that A is normal and thus, A is unitarily diagonalizable, that is, $A = U\text{diag}(\lambda_1, \dots, \lambda_n)U^*$ where U is unitary. From Theorem 2.2.5 b (in the detailed class notes), A must be positive semidefinite.

3. (4 points) Let $A \in M_{m,n}$. Note that for any invertible $P \in M_m$ and any invertible $Q \in M_n$, we have

$$\text{rank}\,(PA) = \text{rank}\,(A) \quad \text{and} \quad \text{rank}\,(AQ) = \text{rank}\,(A)$$

   Let $A = USV^*$ be the singular value decomposition of A. That is $U \in M_m$ is unitary (therefore invertible) and $V \in M_n$ is unitary ($V^* = V^{-1}$). Then

$$\text{rank}\,(A) = \text{rank}\,(USV^*) = \text{rank}\,(SV) = \text{rank}\,(S) = \sum_{j=1}^{k} s_j E_{jj}.$$

But the rank of S is exactly the number of nonzero singular values of S. Note that

$$\text{rank}\,(AA^*) = \text{rank}\,(USS^*U) = \text{rank}\,(SS^*) = \text{rank}\,(S) = \text{rank}\,(A)$$

and

$$\text{rank}\,(A^*A) = \text{rank}\,(V^*S^*SV) = \text{rank}\,(S^*S) = \text{rank}\,(S) = \text{rank}\,(A).$$

4. (4 points) Let $A, B \in M_{m,n}$. Suppose A and B have the same singular values. Then there exists unitary $U_1, U_2 \in M_m$ and unitary $V_1, V_2 \in M_n$ such that

$$A = U_1 S V_1^* \quad \text{and} \quad B = U_2 S V_2^* \quad \Longrightarrow \quad A = \underbrace{(U_1 U_2^*)}_{U}\underbrace{U_2 S V_2^*}_{B}\underbrace{(V_2 V_1^*)}_{V}$$

Note that the product of unitary matrices is also unitary, so $U = U_1U_2^* \in M_m$ and $V_2V_1^* \in M_n$ are unitary.

Conversely, if $A = UBV$ for some unitary $U \in M_m$ and unitary $V \in M_n$. Then AA^* and BB^* are unitarily similar since $AA^* = UBB^*U^*$. Thus, the eigenvalues of AA^* are the same as the eigenvalues of BB^*, as well as their square roots. Therefore, A and B have the same singular values.

5. (8 points) Let $A = A^t \in M_n$ be a complex symmetric matrix. Suppose $u \in \mathbb{C}^n$ is a unit vector such that $|u^tAu|$ is maximum among all unit vectors.

(a) Consider the polar form of the complex number $u^tAu = re^{i\theta}$, where $r \geq 0$. Then $|u^tAu| = r$. Let $v = e^{-i\frac{\theta}{2}}u$. Then $v^t = e^{-i\frac{\theta}{2}}u^t$. Then

$$v^tAv = (e^{-i\frac{\theta}{2}}u^t)A(e^{-i\frac{\theta}{2}}u) = e^{-i\theta}(u^tAu) = e^{-i\theta}(re^{i\theta}) = r = |u^tAu|$$

(b) Suppose $U = \begin{bmatrix} v & W \end{bmatrix}$ is unitary with v in part (a) and $W \in M_{n,n-1}$. WLOG, let us assume $v^tAv > 0$, otherwise, $A = 0$ and the conclusion follows. Let

$$B = [b_{ij}] = U^tAU = \begin{pmatrix} v^tAv & v^tAW \\ W^tAv & W^tAW \end{pmatrix} = \begin{pmatrix} v^tAv & y^t \\ y & A_1 \end{pmatrix},$$

where $y = W^tA^tv = W^tAv$ with $y^t = v^tAW$, and $A_1 = W^tAW = W^tA^tW = A^t$. Let $j \in \{2, \dots, n\}$, and $b_{11} = a > 0$, $b_{1j} = |b_{ij}|e^{ir}$ and $b_{jj}e^{-2ir} = c_1 + ic_2$, where $c_1, c_2 \in \mathbb{R}$. Define

$$x_\theta = (\cos\theta)e_1 + e^{-ir}(\sin\theta)e_j$$

and

$$f(\theta) = |x_\theta^t(U^tAU)x_\theta|^2 = (a\cos^2\theta + 2|b_{1j}|\sin\theta\cos\theta + c_1\sin^2\theta)^2 + (c_2\sin^2\theta)^2,$$

Then

$$f'(\theta) = 2(a\cos^2\theta + 2|b_{1j}|\sin\theta\cos\theta + c_1\sin^2\theta)\Big((-a + c_1)\sin 2\theta + 2|b_{1j}|\cos 2\theta\Big)$$
$$+(4c_2^2\sin^3\theta)\cos\theta,$$

Then $f'(0) = 4a|b_{1j}|$. If $b_{1j} \neq 0$, $f'(0) > 0$ and thus, for sufficiently small $\theta > 0$, $f(\theta) = |(x_\theta^tU^t)A(Ux_\theta)| > f(0) = v^tAv$. Note that $||y|| = ||Ux_\theta|| = ||x_\theta|| = 1$. This contradicts the maximality of v^tAv. Thus $b_{1j} = 0$. This argument holds for $j = 2, \dots, n$. Thus $y^t = 0$ (and so $y = 0$). Thus $B = [v^tAv] \oplus A_1$.

(c) If $n = 1$, then $A = [re^{i\theta}]$, where $r \geq 0$. Then $A = W^t[r]W$, where $W = [e^{i\frac{\theta}{2}}]$.
Suppose that for $A_1 \in M_{n-1}$ satisfying $A_1^t = A_1$, there exists a unitary $W \in M_{n-1}$ such that $A_1 = V^t \text{diag}(r_1, \ldots, r_{n-1})V$, where $r_1, \ldots, r_n \geq 0$.
Let $A \in M_n$ such that $A^t = A$. From (c), we know that there is a unitary U such that $U^tAU = [s] \oplus A_1$, where $A_1^t = A_1$. By the induction hypothesis, there is a unitary V such that

$$U^tAU = [s] \oplus V^t\text{diag}(r_1, \ldots, r_{n-1})V = ([1] \oplus V^t)\text{diag}(s, r_1, \ldots, r_{n-1})([1] \oplus V).$$

Then

$$A = \Big((U^t)^*([1] \oplus V^t)\Big)\text{diag}(s, r_1, \ldots, r_{n-1})\Big(([1] \oplus V)U^*\Big)$$

We can let $W = \Big(([1] \oplus V)U^*\Big)$ so that $W^t = (U^*)^t([1] \oplus V^t) = (U^t)^*([1] \oplus V^t)$ and $(s_1, \ldots, s_n) = (s, r_1, \ldots, r_{n-1})$. Note that

$$WW^* = \Big(([1] \oplus V)U^*)(U\Big(([1] \oplus V^*)\Big) = I = \Big(U([1] \oplus V^*)\Big)\Big(([1] \oplus V)U^*\Big),$$

so that W is unitary.
By the principle of mathematical induction, we get the desired conclusion.

(d) Let $A = A^t = W^t\text{diag}(s_1, \ldots, s_n)W$ as in (a)-(c). Then

$$AA^* = W^t\text{diag}(s_1, \ldots, s_n)WW^*\text{diag}(s_1, \ldots, s_n)(W^t)^* = W^t\text{diag}(s_1^2, \ldots, s_n^2)(W^t)^*.$$

Note that $W^t(W^t)^* = W^t(W^*)^t = (W^*W)^t = I$. Thus, the eigenvalues of AA^* are $s_1^2, \ldots, s_n^2$. Then $s_1, \ldots, s_n \geq 0$ are the singular values of A.

6. (6 points) Let $A \in M_n$ be positive semidefinite with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$.

(a-b) From theorem 2.2.5, there is a unitary $U = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \in M_n$ such that

$$A = U\text{diag}(\lambda_1, \ldots, \lambda_n)U^* = \sum_{j=1}^{n} \lambda_j u_j u_j^*.$$

Note that u_1, u_n are a unit vectors such that $u_1^*Au_1 = \lambda_1$ and $u_n^*Au_n = \lambda_n$. Now, let x be a unit vector in $\mathbb{C}_n$. Note that $y = U^*x = [y_j]$ is also a unit vector, i.e. $\sum_{j=1}^n |y_j|^2 = 1$. Now,

$$x^*Ax = y^*Ay = \sum_{j=1}^{n} \lambda_j |y_j|^2$$

Since $\lambda_1 \geq \lambda_j$, then $\lambda_1|y_j| \geq \lambda_j|y_j|$ for $j = 2, \ldots, n$. Similarly, $\lambda_j|y_j| \geq \lambda_n|y_j|$ for $j = 2, \ldots, n$. Thus

$$\lambda_n = \lambda_n \sum_{j=1}^{n} |y_j|^2 = \sum_{j=1}^{n} \lambda_n|y_j|^2 \leq x^*Ax \leq \sum_{j=1}^{n} \lambda_1|y_j|^2 = \lambda_1 \sum_{j=1}^{n} |y_j|^2 = \lambda_1$$

Therefore,

$$\lambda_1 = u_1^*Au_1 = \max\{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}$$

and

$$\lambda_n = u_n^*Au_n \min\{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}$$

7. (6 points) Let $A \in M_{m,n}$ with nonzero singular values $s_1 \geq \cdots \geq s_k > 0$. Note that

$$||Av|| = \sqrt{(Av)^*(Av)} = \sqrt{(v^*A^*)(Av)} = \sqrt{v^*(A^*A)v}$$

and

$$||A^*u|| = \sqrt{(A^*u)^*(A^*u)} = \sqrt{(u^*A)(A^*u)} = \sqrt{u^*(AA^*)u}$$

Note that for any function f that is positive on a set S, we have $\max_{x\in S}\sqrt{f(x)} = \sqrt{\max_{x\in S} f(x)}$. Thus From problem 6, we have

$$\max\{\|Av\| : v \in \mathbb{C}^n, \|v\| = 1\} = \sqrt{\max\{v^*(A^*A)v : v \in \mathbb{C}^n, \|v\| = 1\}} = s_1$$

and

$$\max\{\|A^*u\| : v \in \mathbb{C}^n, \|v\| = 1\} = \sqrt{\max\{u^*(AA^*)u : u \in \mathbb{C}^n, \|u\| = 1\}} = s_1$$

Similarly, for any function f that is positive on a set S, we have $\min_{x\in S}\sqrt{f(x)} = \sqrt{\min_{x\in S} f(x)}$. Thus From problem 6, we have

$$\min\{\|Av\| : v \in \mathbb{C}^n, \|v\| = 1\} = \sqrt{\min\{v^*(A^*A)v : v \in \mathbb{C}^n, \|v\| = 1\}} = \begin{cases} s_k & \text{if } n = k < \min\{m,n\} \\ 0 & \text{if } k < n \end{cases}$$

and

$$\min\{\|A^*u\| : v \in \mathbb{C}^n, \|v\| = 1\} = \sqrt{\min\{u^*(AA^*)u : u \in \mathbb{C}^n, \|u\| = 1\}} = \begin{cases} s_k & \text{if } m = k < \min\{m,n\} \\ 0 & \text{if } k < m \end{cases}$$

8. (Extra credit, 4 points) Suppose $A = -A^t$ is skew-symmetric and $u_1, u_2 \in \mathbb{C}^n$ are orthonormal pairs such that $|u_1^t A u_2|$ is maximum. Let $U = \begin{bmatrix} u_1 & u_2 & W \end{bmatrix}$ be a unitary unitary matrix where $W \in M_{n,n-2}$. Then

$$U^tAU = \begin{pmatrix} u_1^tAu_1 & u_1^tAu_2 & u_1^tAW \\ u_2^tAu_1 & u_2^tAu_2 & u_2^tAW \\ W^tAu_1 & W^tAu_2 & W^tAW \end{pmatrix} = \begin{pmatrix} a & b & y^t \\ c & d & z^t \\ w & x & A_1 \end{pmatrix}$$

Since $A = -A^t$, then

$$U^tAU = U^t(-A^t)U = \begin{pmatrix} -a & -c & -w^t \\ -b & -d & -x^t \\ -y & -z & -A_1^t \end{pmatrix} \Longrightarrow U^tAU = \begin{bmatrix} 0 & b & y^t \\ -b & 0 & z^t \\ -y & -z & A_1 \end{bmatrix}$$

Therefore $a = d = 0$, $c = -b$ and $A_1 = -A_1^t$. It remains to be show that $y = -w = 0 \in \mathbb{C}^{n-2}$ and $z = -x = 0 \in \mathbb{C}^{n-2}$. Now, let $B = U^tAU = (b_{ij})$ and $b_{12} = b = |b|e^{i\theta}$ and $j \in \{3,\ldots,n\}$. If $b_{1j} = |b_{1j}|e^{i\theta_j}$, define

$$v_\theta = U(\cos\theta e^{-i\theta}e_2 + \sin\theta e^{-i\theta_j}e_j) = \cos\theta e^{-i\theta}u_2 + \sin\theta e^{-i\theta_j}u_j,$$

which is a unit vector in $\mathbb{C}^n$. Define

$$f(\theta) = |u_1^tAv_\theta|^2 = (\cos\theta|b_{12}| + \sin\theta|b_{1j}|)^2$$

$$\Longrightarrow f'(\theta) = 2(\cos\theta|b_{12}| + \sin\theta|b_{1j}|)(-\sin\theta|b_{12}| + \cos\theta|b_{1j}|) \Longrightarrow f'(0) = 2|b_{12}||b_{1j}|$$

If $b_{1j} \neq 0$, then $f'(0) > 0$ and hence there is a sufficiently small θ such that $|u_1^t A v_\theta| > |u_1^t A v_0| = |u_1^t A u_2|$. This contradicts the maximality of $|u_1^t A u_2|$. Thus, $b_{1j} = 0$ for $j = 3, \ldots, n$. That is $y^t = 0$.

Similarly, let $j \in \{3, \ldots, n\}$. If $b_{j2} = |b_{j2}|e^{i\beta_j}$, define

$$w_\theta = U(\cos\theta e^{-i\theta} e_1 + \sin\theta e^{-i\beta} e_j) = \cos\theta e^{-i\theta} u_1 + \sin\theta e^{-i\beta} u_j,$$

which is a unit vector in $\mathbb{C}^n$. Let

$$g(\theta) = |w_\theta^t A u_2|^2 = (\cos\theta |b_{12}| + \sin\theta |b_{j2}|)^2$$

$$\implies g'(\theta) = 2(\cos\theta|b_{12}| + \sin\theta|b_{1j}|)(-\sin\theta|b_{12}| + \cos\theta|b_{j2}|) \implies g'(0) = 2|b_{12}||b_{j2}|$$

If $b_{j2} \neq 0$, then $g'(0) > 0$ and hence there is a sufficiently small θ such that $|w_\theta^t A u_2| > |w_0^t A v_0| = |u_1^t A u_2|$. This contradicts the maximality of $|u_1^t A u_2|$. Thus, $b_{j2} = 0$ for $j = 3, \ldots, n$. That is $-z = 0$. This completes the proof.