Math 408 Advanced Linear Algebra Homework 5 Your Name

1. (8 points) Let $\mathbf{d} = (6,3,1)^t$ and $\mathbf{a} = (8,2,0)^t$

   (a) Find doubly stochastic matrices T_1 and T_2 such that $T_1\mathbf{a} = (6,4,0)^t$ and $T_2T_1\mathbf{a} = \underline{\mathbf{d}}$.

   (b) Construct a real symmetric matrix with eigenvalues $8,2,0$ and diagonal entries $1,3,6$ (in the specified order).

2. (8 points) Let $\mathbf{x} = (x_1,\dots,x_n)^t, \mathbf{y} = (y_1,\dots,y_n) \in \mathbb{R}^n$ with positive entries.

   (a) Suppose $\mathbf{x}$ is obtained from $\mathbf{y}$ by changing two of its entries $y_p > y_q$ to $y_p - d, y_q + d$ with $d \in (0, y_p - y_q)$. Show that $y_py_q \le x_px_q$, and hence $\prod_{j=1}^n y_j \le \prod_{j=1}^n x_j$.

   For example, if $\mathbf{y} = (3,1)$ and $\mathbf{x} = (2,2)$, then $(3)(1) \le (2)(2)$.

   (b) Suppose $\mathbf{x} \prec \mathbf{y}$. Sow that $\prod_{j=1}^n y_j \le \prod_{j=1}^n x_j$.

3. (8 points) (a) Suppose $A = (a_{ij}) \in M_n$ is positive semidefinite. Show that $\det(A) \le \prod_{j=1}^n a_{jj}$.

   Hint: $\det(A) = \prod_{j=1} \lambda_j(A)$, and use the result of the last problem.

   (b) Suppose $B \in M_n$ has columns $b_1,\dots,b_n$. Show $|\det(B)| \le \prod_{j=1}^n \|b_j\|$.

   Hint: Show that $\det(B^*B) = |\det(B)|^2$, B^*B has diagonal entries $\|b_1\|^2,\dots,\|b_n\|^2$, ...

4. (6 points) Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in M_n$ be positive semidefinite with $A_{11} \in M_k$. Show that

$$\det(A) \le \det(A_{11})\det(A_{22}).$$

5. (8 points) Let $A = A^* \in M_n$. Show that there is an invertible matrix S such that $S^*AS = I_p \oplus -I_q \oplus 0_{n-p-q}$ if and only if A has p positive and q negative eigenvalues.

   Hint: To prove the sufficiency, let $U^*AU = \operatorname{diag}(a_1,\dots,a_p,-a_{p+1},\dots,-a_{p+q},0,\dots,0)$ such that $a_1,\dots,a_{p+1} > 0$. Then ...

   Hint: To prove the necessity, let S_1 be the matrix form by the first p columns of S. Apply Gram-Schmidt process so that $S_1 = VR$ for $V \in M_{n,p}, R \in M_p$ with $V^*V = I_p$ and R is invertible upper triangular. Show that $\lambda_k(V^*AV) = \lambda_k((R^*)^{-1}R^{-1}) > 0$ so that A has at least p positive eigenvalues. Similarly, argue that $-A$ has at least q positive eigenvalues, and that A has at least $n-p-q$ zero eigenvalues.

6. (8 points) Suppose $A, B \in M_n$ are Hermitian matrices with eigenvalues $a_1 \ge \cdots \ge a_n$ and $b_1 \ge \cdots \ge b_n$, respectively. Use the result of Lidskii or otherwise to show that

$$(a_1 + b_n, \dots, a_n + b_1) \prec \lambda(A+B) \prec (a_1 + b_1, \dots, a_n + b_n).$$