Math 408 Advanced Linear Algebra Homework 5 Sample Solution

1. (8 points) Let $\mathbf{d} = (6, 3, 1)^t$ and $\mathbf{a} = (8, 2, 0)^t$

   (a) Solving $a \in [0, 1]$ such that $\begin{pmatrix} a & 1-a \\ 1-a & a \end{pmatrix} \begin{pmatrix} 8 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$, we have $a = 2/3$ so that $T_1 = \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix} \oplus [1]$ satisfies $T_1(8, 2, 0)^t = (6, 4, 0)^t$. Similarly, $\begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ so that $T_2 = [1] \oplus \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} (6, 4, 0)^t = (6, 3, 1)^t$. Hence, $T_2 T_1 \mathbf{a} = \mathbf{b}$.

   (b) Suppose $U_1 = \begin{pmatrix} \sqrt{2/3} & -\sqrt{1/3} \\ \sqrt{1/3} & \sqrt{2/3} \end{pmatrix} \oplus [1]$. Then $U_1 \text{diag}\,(8, 2, 0) U_1^* = \begin{pmatrix} 6 & 2\sqrt{2} \\ 2\sqrt{2} & 4 \end{pmatrix} \oplus [0]$. Let $U_2 = [1] \oplus \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$. Then

$$U_2 U_1 \text{diag}\,(8, 2, 0) U_1^* U_2^* = U_2 \begin{pmatrix} 6 & -2\sqrt{2} & 0 \\ -2\sqrt{2} & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} U^* = \begin{pmatrix} 6 & \sqrt{6} & \sqrt{2} \\ \sqrt{6} & 3 & \sqrt{3} \\ \sqrt{2} & \sqrt{3} & 1 \end{pmatrix}.$$

   Note that U_1, U_2 are orthogonal and so is $U = U_2 U_1$. So, for $P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, we get the matrix

$$\boxed{A = PU\text{diag}(8, 2, 0) U^t P^t = \begin{pmatrix} 1 & \sqrt{3} & \sqrt{2} \\ \sqrt{3} & 3 & \sqrt{6} \\ \sqrt{2} & \sqrt{6} & 6 \end{pmatrix}}$$

   satisfying the desired conditions (i.e. real symmetric, has eigenvalues $(8, 2, 0)$ and diagonal entries $(1, 3, 6)$.

2. (8 points) Let $\mathbf{x} = (x_1, \dots, x_n)^t, \mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$ with positive entries.

   (a) Suppose $\mathbf{x}$ is obtained from $\mathbf{y}$ by changing two of its entries $y_p > y_q$ to $y_p - d, y_q + d$ with $d \in (0, y_p - y_q)$. That is,

$$x_j = \begin{cases} y_j & \text{if } j \notin \{p, q\} \\ y_p - d & \text{if } j = p \\ y_q + d & \text{if } j = q \end{cases}$$

   Note that $d, y_p - y_q - d \geq 0$. Then

$$x_p x_q = (y_p - d)(y_q + d) = y_p y_q + d(y_p - y_q - d) \geq y_p y_q.$$

   Hence $\prod_{j=1}^n x_j = x_p x_q \prod_{j=1, j\neq p,q}^n x_j = x_p x_q \prod_{j=1, j\neq p,q}^n y_j \geq \prod_{j=1}^n y_j$

   (b) Suppose $\mathbf{x} \prec \mathbf{y}$. Then there exists a sequence $(\mathbf{y}_0, \mathbf{y}_1, \dots, \mathbf{y}_{k+1})$ such that $\mathbf{y}_0 = \mathbf{y}$, $\mathbf{y}_{k+1} = \mathbf{x}$ and $\mathbf{y}_{r+1} = (y_1^{(r+1)}, \dots, y_n^{(r+1)})$ is obtained from $\mathbf{y}_r = (y_1^{(r)}, \dots, y_n^{(r)})$ by changing two of its entries as described in problem (a). Thus for any $r = 0, \dots, k$, we have

$$\prod_{j=1}^n y_j^{(r)} \leq \prod_{j=1}^n y_j^{(r+1)} \quad \Longrightarrow \quad \prod_{j=1}^n y_j = \prod_{j=1}^n y_j^{(0)} \leq \prod_{j=1}^n y_j^{(k+1)} = \prod_{j=1}^n x_j$$

3. (8 points) (a) Suppose $A = (a_{ij}) \in M_n$ is positive semidefinite. Let $\lambda = (\lambda_1(A), \ldots, \lambda_n(A))^t$ be the vector of eigenvalues of A and $d = (a_{11}, \ldots, a_{nn})^t$ be the vector of composed of the diagonal entries of A. By Theorem 3.1.3, $d \prec \lambda$.

If A has zero eigenvalues, then $\det(A) = 0$. Note that all diagonal entries of A must be nonnegative since A is positive semidefinite. Trivially, $0 = \det(A) \leq \prod_{j=1}^{n} a_{jj}$.

Suppose A is positive definite (all eigenvalues are positive). Note that this implies that the diagonal entries of A are all positive (The existence of a zero diagonal entry will imply that the entire column/row of that entry is zero). Applying problem 2b, we get

$$\prod_{j=1} \lambda_j(A) = \det(A) \leq \prod_{j=1}^{n} a_{jj}.$$

(b) Suppose $B \in M_n$ has columns $b_1, \ldots, b_n$.

Note that for any X, $\det(\bar{X}) = \overline{\det(X)}$ because in the determinant expansion, all numbers are replaced by its conjugates. Thus $\det(B^*) = \overline{det(B^t)} = \overline{det(B)}$ and hence

$$\det(B^*B) = \det(B^*)\det(B) = |\det(B)|^2$$

Note that the j^{th} diagonal entry of B^*B is the the product of the j^{th} row of B^* and j^{th} column of B, i.e., $b_j^* b_j = ||b_j||^2$. Since B^*B is positive semidefinite, we can apply Problem 3a to get

$$\det(B^*B) = |\det(B)|^2 \leq \prod_{j=1}^{n} ||b_j||^2$$

Taking the square root of both sides, we get $|\det(B)| \leq \prod_{j=1}^{n} \|b_j\|$.

4. (6 points) Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in M_n$ be positive semidefinite with $A_{11} \in M_k$. Then A_{11}, A_{22} are also positive semidefinite and $A_{21} = A_{12}^*$. Suppose $A_{11} = U_1 D_1 U_1^*$ and $A_{22} = U_2 D_2 U_2^*$, where U_1 and U_2 are unitary and D_1, D_2 are diagonal with nonnegative entries. Then

$$\hat{A} = \begin{pmatrix} U_1^* & 0 \\ 0 & U_2^* \end{pmatrix} A \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} = \begin{pmatrix} D_1 & U_1^* A_{12} U_2 \\ U_2^* A_{12}^* U_1 & D_2 \end{pmatrix}$$

$\hat{A}$ is still positive semidefinite since it is unitarily similar to A. Applying problem 3a to $\hat{A}$, we get

$$\det(\hat{A}) = \det(A) \leq det(D_1)\det(D_2) = \det(A_{11})\det(A_{22})$$

5. (8 points) Let $A = A^* \in M_n$.

($\Longleftarrow$) Suppose A has p positive and q negative eigenvalues. By the spectral decomposition, there exists a unitary U such that

$$U^*AU = \operatorname{diag}(a_1, \ldots, a_p, -a_{p+1}, \ldots, -a_{p+q}, 0, \ldots, 0),$$

where $a_1, \ldots, a_{p+q} > 0$. For $j = 1, \ldots, n$, define

$$d_j = \begin{cases} \frac{1}{\sqrt{a_j}} & \text{if } 1 \leq j \leq p+q \\ 1 & \text{if } j > p+q \end{cases}$$

and define $S = U\text{diag}(d_1, \dots, d_n)$, which is invertible since U is unitary and $d_j > 0$ for all j. Then $S^*AS = I_p \oplus -I_q \oplus 0_{n-p-q}$.

($\Longrightarrow$) Suppose $S^*AS = I_p \oplus -I_q \oplus 0_{n-p-q}$ for some invertible S. Partition $S = \begin{bmatrix} S_1 & S_2 & S_3 \end{bmatrix}$ such that $S_1 \in M_{n,p}, S_2 \in M_{n,q}$ and $S_3 \in M_{n,n-p-q}$. Then

$$\begin{pmatrix} S_1^*AS_1 & S_1^*AS_2 & S_1^*AS_3 \\ S_2^*AS_1 & S_2^*AS_2 & S_2^*AS_3 \\ S_3^*AS_1 & S_3^*AS_2 & S_3^*AS_3 \end{pmatrix} = \begin{pmatrix} I_p & 0 & 0 \\ 0 & -I_q & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

By the QR decomposition theorem, there exists an invertible upper triangular $R_1 \in M_p$ and an isometry $V_1 \in M_{n,p}$, i.e. $V_1^*V_1 = I_p$ such that $S_1 = V_1R_1$. Since $S_1^*AS_1 = I_p$, then $V_1^*AV = (R^*)^{-1}R^{-1}$, which is a positive definite $p \times p$ matrix by Theorem 2.2.5 c. Now, using the min-max characterization of the eigenvalues of A, we have

$$\lambda_p(A) \geq \lambda_p(V_1^*AV_1) = \lambda_p((R^*)^{-1}R^{-1}) > 0$$

Thus, A must have at least p positive eigenvalues.

Similarly, Let $S_2 = V_2R_2$ for some $V_2 \in M_{n,q}$ satisfying $V_2^*V_2 = I_q$ and an invertible upper triangular $R_2 \in M_q$. Since $S_2^*AS_2 = -I_q$, then $V_2^*AV_2 = -(R_2^*)^{-1}R_2^{-1}$ which is negative definite (all eigenvalues are negative).

$$\lambda_{n-q+1}(A) \leq \lambda_1(V_2AV_2^*) = \lambda_1(-(R_2^*)^{-1}R_2^{-1}) < 0$$

Hence $\lambda_{n-q+1}(A), \dots, \lambda_n(A)$ — the last q eigenvalues of A are negative.

Now, since S is invertible, then $\text{rank}\,(A) = \text{rank}\,(S^*AS) = p + q$. Since A hermitian, this means that it is diagonalizable and its rank is equal to the number of its nonzero eigenvalues. Therefore A must have $n-p-q$ zero eigenvalues. This forces the number of positive eigenvalues of A to be exactly p and the number of negative eigenvalues of A to be exactly q.

6. (8 points) Lidskii's inequality (Theorem 3.3.2) states that for any $1 \leq k \leq n$ and $1 \leq r_1 < r_2 < \cdots < r_k \leq n$, it holds that for any $n \times n$ Hermitian matrices X and Y

$$\sum_{j=1}^{k} \lambda_{r_j}(X + Y) \leq \sum_{j=1}^{k} \lambda_{r_j}(X) + \lambda_j(Y)$$

Suppose $A, B \in M_n$ are Hermitian matrices with eigenvalues $a_1 \geq \cdots \geq a_n$ and $b_1 \geq \cdots \geq b_n$, respectively. Let

$$\mathbf{d} = (a_1 + b_n, a_2 + b_{n-1}, \dots, a_n + b_1) \quad \text{and} \quad \mathbf{u} = (a_1 + b_1, \dots, a_n + b_n).$$

(*First, we will show that* $\lambda(A + B) \prec \mathbf{u}$)

Obviously, $\sum_{j=1}^{n} \lambda_j(A + B) = \text{tr}(A + B) = \sum_{j=1}^{n}(a_j + b_j)$. Now, suppose $1 \leq k < n$. If we apply Lidskii's inequality to $(r_1, \dots, r_k) = (1, \dots, k)$, we get

$$\sum_{j=1}^{k} \lambda_j(A + B) \leq \sum_{j=1}^{k} a_j + b_j = \text{ sum of } k \text{ largest entries of } \mathbf{u}$$

This shows $\lambda(A+B) \prec \mathbf{u}$.

(*Next, we will show that* $\mathbf{d} \prec \lambda(A+B)$)
Obviously, $\sum_{j=1}^{n} \lambda_j(A+B) = \sum_{j=1}^{n}(a_j + b_j) = \sum_{j=1}^{n}(a_j + b_{n-j+1})$. Now, suppose $1 \le k < n$. Let $(s_1, \ldots, s_n)$ be the rearragement of $(1, \ldots, n)$ such that

$$a_{s_1} + b_{n-s_1+1} \ge a_{s_2} + b_{n-s_2+1} \ge \cdots \ge a_{s_n} + b_{n-s_n+1}.$$

Let $C = A + B$ so that $A = (-B) + C$. If we apply Lidskii's inequalities (Theorem 3.3.2) with $\{r_1, \ldots, r_k\} = \{s_1, \ldots, s_k\}$

$$\sum_{j=1}^{k} \lambda_{r_j}((-B)+C) \le \sum_{j=1}^{k} \lambda_{r_j}(-B) + \lambda_j(C)$$

The left hand side of this inequality is

$$\sum_{j=1}^{k} \lambda_{r_j}((-B)+C) = \sum_{j=1}^{k} \lambda_{r_j}(A) = \sum_{j=1}^{k} a_{r_j} = \sum_{j=1}^{k} a_{s_j}$$

while the right hand side is

$$\sum_{j=1}^{k} \lambda_{r_j}(-B) + \lambda_j(C) = \sum_{j=1}^{k} -b_{n-r_j+1} + \lambda_j(A+B) = \sum_{j=1}^{k} -b_{n-s_j+1} + \lambda_j(A+B).$$

Thus, $\sum_{j=1}^{k} a_{s_j} \le \sum_{j=1}^{k} -b_{n-r_j+1} + \lambda_j(A+B)$, which implies

$$\sum_{j=1}^{k} \lambda_j(A+B) \ge \sum_{j=1}^{k} a_{s_j} + b_{n-s_j+1} = \text{ sum of } k \text{ largest entries of } \mathbf{d}$$

Therefore, $\mathbf{d} \prec \lambda(A+B)$.