Math 408 Advanced Linear Algebra Homework 6 Your Name

1. (8 points) Construct a real symmetric matrix with eigenvalues $3, 2, 1$ so that the leading 2×2 submatrix has eigenvalues $2.8, 1.5$.

2. (8 points) Let $A^* = A = (a_{ij}) \in M_n$ with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Then $a_{11} + \cdots + a_{kk} = \lambda_1 + \cdots + \lambda_k$ if and only if $A = A_{11} \oplus A_{22}$ such that $A_{11} \in M_k$ has eigenvalues $\lambda_1, \ldots, \lambda_k$.

   Hint: Let $\mathcal{S}(\lambda_1, \ldots, \lambda_n)$ be the set of Hermitian matrices with eigenvalues $\lambda_1, \ldots, \lambda_n$. Suppose $A = (a_{ij}) \in \mathcal{S}(\lambda_1, \ldots, \lambda_n)$ is such that $a_{11} + \cdots + a_{kk}$ is maximum. First prove $A = A_{11} \oplus A_{22}$ as follows. Suppose $a_{ij} \neq 0$ for some a_{ij} with $i \leq k < j$. Show that the submatrix lying in rows i and j is unitarily similar to a matrix with $(1,1)$ entry $\mu > a_{ii}$ Then deduce that A is unitarily similar to $\hat{A}$ with the sum of the first k diagonal entries having a sum larger than that of A to derive a contradiction. After that show that A_{11} has eigenvalues $\lambda_1, \ldots, \lambda_k$.

3. (8 points) Let $A \in M_n$ with singular values $s_1 \geq \cdots \geq s_n$. Show that there is a unitary matrix $U \in M_n$ such that $U^*AU = (a_{ij})$ such that $|a_{11}| + \cdots + |a_{nn}| = s_1 + \cdots + s_n$.

   Hint: Let $A = PV$ where P is positive semidefinite, and V is unitary such that $V = UDU^*$, where U is unitary and D is diagonal.

4. (4 points) A matrix $A \in M_n$ with singular values $s_1 \geq \cdots \geq s_n$ with $s_1 \leq 1$ is called a contraction. Show that A is a contraction if and only if $I \geq AA^*$, i.e., $I - AA^*$ is positive semidefinite.

5. (8 points) For a positive definite matrix $X \in M_n$, let $|X| = \sqrt{X}$ be the (unique) positive semidefinite matrix such that $|X|^2 = X^*X$. Let $A \in M_n$ be a contraction.

   (a) Show that $A\sqrt{I - A^*A} = \sqrt{I - AA^*}A$. (Hint: Let $A = UDV$, the svd, then $I - A^*A = V^*(I - D^2)V$ so that $\sqrt{I - A^*A} = ...$)

   (b) Show that if $A \in M_n$ is a contraction, then $\begin{pmatrix} A & \sqrt{1 - AA^*} \\ \sqrt{I - A^*A} & -A^* \end{pmatrix}$ is unitary.

6. (8 points) Let $A \in M_n$ be a contraction with polar decomposition PV such that P is positive semidefinite and V is unitary.

   (a) Show that $A_1 = (P + i\sqrt{I - P^2})V$ and $A_2 = (P - i\sqrt{I - P^2})V$ are unitary.

   (b) Show that $A = (A_1 + A_2)/2$.

   Note that if $n = 1$, this shows that every complex number μ with $|\mu| \leq 1$ is the average of two complex numbers of unit moduli.

7. (8 points) Let $A \in M_n$ be a nonzero positive semidefinite matrix. Show that $tr\, A^2 \leq (\mathrm{tr}\, A)^2$; the equality holds if and only if A has rank one.

8. (Extra 8 points) Let $(a_1, \ldots, a_n), (b_1, \ldots, b_n)$ be real vectors. Show that there is $A \in M_n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ satisfying $(a_1, \ldots, a_n) = (\lambda_1 + \bar{\lambda}_1, \ldots, \lambda_n + \bar{\lambda}_n)$ such that $A + A^*$ has eigenvalues $b_1, \ldots, b_n$ if and only if $(a_1, \ldots, a_n) \prec (b_1, \ldots, b_n)$.

   Hint: Suppose U^*AU is in triangular form. Consider the diagonal entries and eigenvalues of $U^*(A+A^*)U$.... Conversely, let H be a Hermitian matrix with diagonal entries $a_1, \ldots, a_n$ and eigenvalues $b_1, \ldots, b_n$. Construct $G = G^*$ such that $A = H + iG$ has the desired eigenvalues and diagonal entries. (Think about a 2×2 matrix.)