1. (8 points) From the proof of Theorem 3.4.2, we take $(c_1, c_2, c_3) = (3, 2, 1)$ and $(a_1, a_2) = (2.8, 1.5)$ and define u_1, u_2, u_3 as follows

$$u_1 = \sqrt{\tfrac{(c_1-a_1)(c_1-a_2)}{(c_1-c_2)(c_1-c_3)}} = \sqrt{0.15},\ u_2 = \sqrt{\tfrac{(c_2-a_1)(c_2-a_2)}{(c_2-c_1)(c_2-c_3)}} = \sqrt{0.4},\ u_3 = \sqrt{\tfrac{(c_3-a_1)(c_3-a_2)}{(c_3-c_1)(c_3-c_2)}} = \sqrt{0.45}$$

Let $u = \begin{bmatrix}\sqrt{0.15} & \sqrt{0.4} & \sqrt{0.45}\end{bmatrix}^T$. We can apply the Gram-Schmidt process on $\{u, e_2, e_3\}$ to get $\{u, w_1, w_2\}$ which will be an orthonormal basis for $\mathbb{R}^3$. Then the following matrix is real orthogonal

$$Q = \begin{bmatrix} w_1 & w_2 & u \end{bmatrix} = \begin{bmatrix} \sqrt{0.75} & \sqrt{0.1} & \sqrt{0.15} \\ 0 & -\sqrt{0.6} & \sqrt{0.4} \\ -0.5 & \sqrt{0.3} & \sqrt{0.45} \end{bmatrix}.$$

Let

$$A = Q^t \operatorname{diag}(3,2,1) Q = \boxed{\begin{bmatrix} 2.5 & \sqrt{0.3} & \sqrt{0.45} \\ \sqrt{0.3} & 1.8 & -\sqrt{1.5} \\ \sqrt{0.45} & -\sqrt{1.5} & 1.7 \end{bmatrix}}$$

One easily checks that $\begin{bmatrix} 2.5 & \sqrt{0.3} \\ \sqrt{0.3} & 1.8 \end{bmatrix}$ has eigenvalues 2.8 and 1.5.

2. (8 points) Let $\mathcal{S}(\lambda_1, \dots, \lambda_n)$ be the set of all $n \times n$ Hermitian matrices with eigenvalues $(\lambda_1, \dots, \lambda_n)$, which is a compact set. Note that for any $B = (b_{ij}) \in \mathcal{S}$, $(b_{11}, \dots, b_{nn}) \prec (\lambda_1, \dots, \lambda_n)$. Thus for any $k = 1, \dots, n$,

$$b_{11} + \cdots + b_{kk} \le \ k \text{ largest diagonal entries of } B \le \lambda_1 + \cdots + \lambda_k.$$

Let $A^* = A = (a_{ij}) \in M_n$ with eigenvalues $\lambda_1 \ge \cdots \ge \lambda_n$.
($\Longleftarrow$) If $A = A_{11} \oplus A_{22}$ where $A_{11} \in M_k$ has eigenvalues $\lambda_1, \dots, \lambda_k$. Then $a_{11} + \cdots + a_{kk} = tr(A_{11}) = \lambda_1 + \cdots + \lambda_k$.

($\Longrightarrow$) Suppose $a_{11} + \cdots + a_{kk} = \lambda_1 + \cdots + \lambda_k$. Then applying the claim below, we get that $A = A_{11} \oplus A_{22}$ where $A_{11} \in M_k$. Note that the eigenvalues of A are the eigenvalues of A_{11} together with the eigenvalues of A_{22}. Thus, it must be true that the eigenvalues of A_{11} are $\lambda_1, \dots, \lambda_k$.
Claim: Suppose $A = (a_{ij}) \in \mathcal{S}$ such that for any $B = (b_{ij}) \in \mathcal{S}$, it holds that $\sum_{s=1}^k b_{ss} \le \sum_{s=1}^k a_{ss}$. Then $a_{ij} = 0 = \overline{a_{ji}}$ whenever $i \le k < j$. (Equivalently, $A = A_{11} \oplus A_{22}$ for some $A_{11} \in M_k$.)
Proof of Claim: Suppose otherwise. Then there exist (i, j) with $i \le k < j$ such that $a_{ij} \neq 0$. Let $C = \begin{bmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{bmatrix}$ and γ_1, γ_2 be its eigenvalues. Note that C is Hermitian since it is a principal submatrix of A. Let $\hat{U}$ be the unitary matrix such that $\hat{U}C\hat{U}^* = \operatorname{diag}(\gamma_1, \gamma_2)$. We know from Rayleigh's theorem that $\lambda_1 \ge a_{ii}$. Note, in fact that $\lambda_1 > a_{ii}$ because if it were true that $\lambda_1 = a_{ii}$, then $\lambda_2 = a_{jj}$ by comparing the trace of C and $\hat{U}C\hat{U}^*$. But from Problem 5 and 6a of Homework 3, we know that the sum of the singular values of C are given by

$$\lambda_1^2 + \lambda_2^2 = a_{ii}^2 + a_{jj}^2 + 2|a_{ij}|^2 > a_{ii}^2 + a_{jj}^2.$$

Therefore $\lambda_1 > a_{ii}$. Now Let $B = UAU^* = (b_{ij})$ where the entries of $U = (u_{ij})$ is the same as that of the identity except for four positions–specifically,

$$\begin{bmatrix} u_{ii} & u_{ij} \\ u_{ji} & u_{jj} \end{bmatrix} = \hat{U}$$

Then the entries of A and B are the same except for the i^{th} and j^{th} rows and columns. Now,

$$\sum_{s=1}^{k} b_{ss} = b_{ii} + \sum_{s=1, s\neq i}^{k} b_{ss} = \gamma_1 + \sum_{s=1, s\neq i}^{k} a_{ss} > \sum_{j=1}^{k} a_{jj}$$

which is a contradiction to the assumption of the claim. Therefore $a_{ij} = 0 = a_{ji}$ for any $i \leq k < j$.

3. (8 points) Let $A \in M_n$ with singular values $s_1 \geq \cdots \geq s_n$. By the Corollary 2.4.4. there is a positive semidefinite matrix P with eigenvalues $s_1 \geq \cdots \geq s_n$ and a unitary V such that $A = PV$. Since V is unitary, then it is normal and its eigenvalues are unit complex numbers. Thus $V = UDU^*$, where U is unitary and is a diagonal matrix of the form $D = \text{diag}(e^{i\theta_1}, \dots, e^{i\theta_n})$. Then

$$U^*AU = U^*(PUDU^*)U = U^*PUD$$

Now, suppose $U^*AU = (a_{ij})$ and $B = U^*PU = (b_{ij})$. Notice that B is unitarily similar to P. Thus, B is also positive semidefinite. Furthermore, note that $a_{jj} = e^{i\theta_j} b_{jj}$ for $j = 1, \dots, n$. Thus,

$$|a_{11}| + \cdots + |a_{nn}| = b_{11} + \cdots + b_{nn} = tr(B) = tr(P) = s_1 + \cdots s_n$$

4. (4 points) Suppose $A \in M_n$ has singular values $s_1 \geq \cdots \geq s_n$. Then AA^* has eigenvalues $s_1^2 \geq s_2^2 \geq \cdots \geq s_n^2$.

If A is a contraction, then $s_1^2 \leq 1$. By the Rayleigh principle, for any unit vector $x \in \mathbb{C}^n$,

$$x^*AA^*x \leq s_1^2 \quad \Longrightarrow \quad x^*(I - AA^*)x = 1 - x^*AA^*x \geq 1 - s_1^2 \geq 0$$

Thus, $I - AA^*$ is positive semidefinite, i.e. $I \geq AA^*$.

Conversely, If $I \geq AA^*$ then for any unit vector $x \in \mathbb{C}^n$, $x^*(I - AA^*)x = 1 - x^*(AA^*)x \geq 0$. In particular, if x is a unit eigenvector of AA^* corresponding to s_1^2, then this implies $1 - s_1^2 \geq 0$. Then $s_1 \leq 1$ and A is a contraction.

5. (8 points) For a positive semidefinite matrix $X \in M_n$, let $|X| = \sqrt{X}$ be the (unique) positive semidefinite matrix such that $|X|^2$. Let $A \in M_n$ be a contraction.

(a) Consider the singular value decomposition of A given by $A = UDV$. Then

$$I - A^*A = I - V^*D^2V = V^*(I - D^2)V \quad \text{and} \quad I - AA^* = I - UD^2U^* = U(I - D^2)U^*$$

Since A is a contraction, then $I - D^2$ is a diagonal matrix with nonnegative diagonal entries.

$$A\sqrt{I - A^*A} = (UDV)(V^*\sqrt{I - D^2}V) = U(D\sqrt{I - D^2})V$$

Here, $\sqrt{I - D^2}$ is the diagonal matrix whose diagonal entry is just the square roots of the corresponding diagonal entries of $I - D^2$. Since D and $\sqrt{I - D^2}$ are diagonal matrices, they commute and thus,

$$A\sqrt{I - A^*A} = U(\sqrt{I - D^2})DV = (U(\sqrt{I - D^2})U^*)(UDV) = \sqrt{1 - AA^*}A$$

(b) Suppose $A \in M_n$ is a contraction. Then $\sqrt{1 - AA^*}$ and $\sqrt{I - A^*A}$ are well-defined psd (therefore Hermitian) matrices. Let

$$U = \begin{pmatrix} A & \sqrt{1 - AA^*} \\ \sqrt{I - A^*A} & -A^* \end{pmatrix} \in M_{2n}.$$

Then

$$\begin{aligned} UU^* &= \begin{pmatrix} A & \sqrt{1 - AA^*} \\ \sqrt{I - A^*A} & -A^* \end{pmatrix} \begin{pmatrix} A^* & \sqrt{I - A^*A} \\ \sqrt{I - AA^*} & -A \end{pmatrix} \\ &= \begin{pmatrix} AA^* + (\sqrt{I - AA^*})^2 & A\sqrt{I - A^*A} - \sqrt{I - AA^*}A \\ \sqrt{I - A^*A}A^* - A^*\sqrt{I - AA^*} & (\sqrt{I - A^*A})^2 + A^*A \end{pmatrix} \\ &= \begin{pmatrix} I & A\sqrt{I - A^*A} - \sqrt{I - AA^*}A \\ (A\sqrt{I - A^*A} - \sqrt{I - AA^*}A)^* & I \end{pmatrix} \end{aligned}$$

By Part (a), $A\sqrt{I - A^*A} - \sqrt{I - AA^*}A = 0$ and thus, U is unitary.

6. (8 points) Let $A \in M_n$ be a contraction with polar decomposition PV such that P is positive semidefinite and V is unitary.

(a) The eigenvalues of P are the singular values of A. Since A is contraction, then $I \geq P^2$ and hence $\sqrt{I - P^2}$ is well-defined. Note also that P and $\sqrt{I - P^2}$ are psd, and hence Hermitian, so that $P = P^*$ and $(\sqrt{I - P^2})^* = \sqrt{I - P^2}$. Let $Q = P + i\sqrt{I - P^2} \in M_n$. Then

$$\begin{aligned} QQ^* &= (P + i\sqrt{I - P^2})(P + i\sqrt{I - P^2})^* \\ &= (P + i\sqrt{I - P^2})(P - i\sqrt{I - P^2}) \\ &= P^2 + iP\sqrt{I - P^2} - iP\sqrt{I - P^2} + (\sqrt{I - P^2})^2 \\ &= I \end{aligned}$$

Hence Q is unitary, and therefore Q^* is also unitary. Define $A_1 = (P + i\sqrt{I - P^2})V = QV$ and $A_2 = (P - i\sqrt{I - P^2})V = Q^*V$. Clearly A_1 and A_2 are unitary since the product of unitary matrices is unitary.

(b) Now

$$\begin{aligned} \frac{1}{2}(A_1 + A_2) &= \tfrac{1}{2}\Big((P + i\sqrt{I - P^2})V + (P - i\sqrt{I - P^2})V\Big) \\ &= \tfrac{1}{2}\Big((P + i\sqrt{I - P^2}) + (P - i\sqrt{I - P^2})\Big)V = \tfrac{1}{2}(2P)V = PV = A \end{aligned}$$

Note that if $n = 1$, this shows that every complex number μ with $|\mu| \leq 1$ is the average of two complex numbers of unit moduli.

7. (8 points) Let $A = (a_{ij})$ be a positive semidefinite matrix with eigenvalues $s_1 \geq \cdots \geq s_n \geq 0$. Note that the eigenvalues of A^2 are $s_1^2, \ldots, s_n^2$. Thus,

$$\text{tr}\,(A^2) = s_1^2 + \cdots + s_n^2$$

while

$$(\text{tr}\,A)^2 = (s_1 + \cdots + s_n)^2 = s_1^2 + s_2^2 + \cdots + s_n^2 + \sum_{1 \leq i < j \leq n} 2s_is_j \geq \text{tr}\,(A^2)$$

Equality holds if $\sum_{1 \leq i < j \leq n} 2s_is_j = 0$. Since A is nonzero, then $s_1 > 0$. Thus $s_2, \ldots, s_n = 0$, i.e. A must be rank one.

8. (Extra 8 points) Let $(a_1, \ldots, a_n), (b_1, \ldots, b_n)$ be real vectors. Show that there is $A \in M_n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ satisfying $(a_1, \ldots, a_n) = (\lambda_1 + \bar{\lambda}_1, \ldots, \lambda_n + \bar{\lambda}_n)$ such that $A + A^*$ has eigenvalues $b_1, \ldots, b_n$ if and only if $(a_1, \ldots, a_n) \prec (b_1, \ldots, b_n)$.

   Hint: Suppose U^*AU is in triangular form. Consider the diagonal entries and eigenvalues of $U^*(A+A^*)U$.... Conversely, let H be a Hermitian matrix with diagonal entries $a_1, \ldots, a_n$ and eigenvalues $b_1, \ldots, b_n$. Construct $G = G^*$ such that $A = H + iG$ has the desired eigenvalues and diagonal entries. (Think about a 2×2 matrix.)

   Proof:
   Let $(a_1, \ldots, a_n), (b_1, \ldots, b_n)$ be real vectors.

   $(\Longleftarrow)$ Suppose $(a_1, \ldots, a_n) \prec (b_1, \ldots, b_n)$. By Theorem 3.1.3, there is a Hermitian matrix $H = (h_{ij})$ with diagonal entries $(a_1, \ldots, a_n)$ and eigenvalues $b_1, \ldots, b_n$. Take $\lambda_j = \frac{a_j}{2} = \overline{\lambda_j}$. Let A and B be the unique pair of upper and lower triangular matrices with diagonal entries $(\frac{a_1}{2}, \ldots, \frac{a_n}{2})$ satisfying $H = A + B$. In fact, $B = A^*$ since H is Hermitian.

   $(\Longleftarrow)$ Conversely, suppose $A \in M_n$ satisfying the following three conditions

   (a) A has eigenvalues $\lambda_1, \ldots, \lambda_n$,

   (b) $a_j = \lambda_j + \overline{\lambda_j}$ for $j = 1, \ldots, n$ and;

   (c) $A + A^*$ has eigenvalues $(b_1, \ldots, b_n)$.

   By Schur's triangularization theorem, there exists a unitary U such that $T = UAU^*$ is upper triangular with diagonal entries $\lambda_1, \ldots, \lambda_n$. Thus, $T + T^* = U(A + A^*)U^*$ has diagonal entries $a_1, \ldots, a_n$ and eigenvalues $b_1, \ldots, b_n$. By Theorem 3.1.3, $(a_1, \ldots, a_n) \prec (b_1, \ldots, b_n)$.