Math 408 Advanced Linear Algebra Homework 6 Your Name

Eight points for each question.

1. Suppose $n = 3$. List all the Horn's sequences $(u_1, u_2), (v_1, v_2), (w_1, w_2)$ of length 2, and list all the Thompson standard sequences $(u_1, u_2), (v_1, v_2)$ and $(w_1, w_2) = (u_1 + v_1 - 1, u_2 + v_2 - 2)$.

   Hint: See p.23-24 in http://cklixx.people.wm.edu/teaching/math408/note.pdf

   You should see six sets of Horn's sequences $(u_1, u_2), (v_1, v_2), (w_1, w_2)$, and one of them not Thompson standard sequences.

2. Let $A, B, C = A + B \in M_n$ be Hermitian with eigenvalues $a_1 \geq \cdots \geq a_n$, $b_1 \geq \cdots \geq b_n$ and $c_1 \geq \cdots \geq c_n$, respectively. Show that if $C = (c_{ij})$ then

$$\sum_{j=1}^{k} c_{jj} \leq \sum_{j=1}^{k} (a_j + b_j);$$

   the equality holds if and only if $A = A_{11} \oplus A_{22}, B = B_{11} \oplus B_{22}$ with $A_{11}, B_{11} \in M_k$ such that A_{11} and B_{11} have eigenvalues $a_1 \geq \cdots \geq a_k, b_1 \geq \cdots \geq b_k$, respectively.

3. (Weyl's inequalities.) Suppose $A, B, C = A+B \in M_n$ are Hermitian matrices. Let $\{x_1, \ldots, x_n\}$, $\{y_1, \ldots, y_n\}$, $\{z_1, \ldots, z_n\}$ be orthonormal sets such that $Ax_j = \lambda_j(A)x_j, By_j = \lambda_j(B)y_j, Cz_j = \lambda_j(C)z_j$. Suppose i, j are positive integers such that $i + j - 1 \leq n$.

   (a) Show that there is a unit vector $v \in V_1 \cap V_2 \cap V_3$, where

   $V_1 = \operatorname{span}\{x_i, \cdots, x_n\}$, $V_2 = \operatorname{span}\{y_j, \ldots, y_n\}$ and $V_3 = \{z_1, \ldots, z_{i+j-1}\}$.

   Hint: Show that $V_2 \cap V_3$ has dimension at least $(n-j+1)+(i+j-1)-n = i$ by considering the null space of the matrix $[y_j \cdots y_n \; z_1 \ldots z_{i+j-1}]$, and then show that $V_1 \cap (V_2 \cap V_3)$ is not the zero space.

   (b) Show that the vector v found in (a) satisfies

$$\lambda_{i+j-1}(C) \leq v^*Cv, \; v^*Av \leq \lambda_i(A), \; v^*Bv \leq \lambda_j(B),$$

   and deduce that

$$\lambda_{i+j-1}(C) \leq \lambda_i(A) + \lambda_j(B).$$

4. Show that $\ell_p(v) \geq \ell_q(v)$ for any vector $v \in \mathbb{F}^n$ if $1 \leq p \leq q \leq \infty$.

   Hint: Only need to prove the result for vector with nonnegative entries; show that $f(v) = \sum v_j^p \geq 1$ if $g(v) = \sum v_j^q - 1 = 0$. Use Lagrange multipliers to the function $L(\mu, v) = f(v) - \mu g(v)$ and conclude that all nonzero entries of v have to be the same so that $v = \gamma(1, \ldots, 1, 0, \ldots, 0)$.

5. Show that if $x, y \in \mathbb{R}^n$ are vectors with positive entries such that $x \prec y$, then $\ell_p(x) \leq \ell_p(y)$ for any $p \geq 1$.

   Hint: We need only do the special case when x is obtained from y by changing two entries $y_i > y_j$ to $y_i - d, y_j + d$ for $d \in (0, y_i - y_j)$.

6. Let $u = (u_1, \ldots, u_n)^t$ and $v = (v_1, \ldots, v_n)^t$ be such that $(|u_1|, \ldots, |u_n|) \prec_w (|v_1|, \ldots, |v_n|)$. Show that there is a nonnegative integer m and a nonnegative d such that

$$(|u_1|, \ldots, |u_n|, \underbrace{d, \ldots, d}_{m}) \prec (|v_1|, \ldots, |v_n|, 0, \ldots, 0).$$

Deduce from the result of the previous problem that $\ell_p(u) \le \ell_p(v)$ for any $p \ge 1$.

(Extra credits) Alternatively, show that there is $\hat{v} = (\hat{v}_1, \ldots, \hat{v}_n)$ such that $\sum_{j=1}^k \hat{v}_j = \max\{\sum_{j=1}^k |v_j|, \ell_1(u)\}$. Then prove that $(|u_1|, \ldots, |u_n|) \prec (\hat{v}_1, \ldots, \hat{v}_n)$ and

$$\ell_p(u) \le \ell_p(\hat{v}) \le \ell_p(v).$$

7. (Extra credits) Suppose $c_1 \ge a_1 \ge c_2 \ge a_2 \ge \cdots \ge a_{n-1} \ge c_n \ge a_n$ are $2n$ real numbers. Show that there is a nonnegative real vector $v \in \mathbb{R}^n$ such that $D + vv^t$ has eigenvalues $c_1 \ge \cdots \ge c_n$ for $D = \mathrm{diag}\,(a_1, \ldots, a_n)$.

Hint: Replace c_j by $c_j + \gamma$ and $a_j + \gamma$ for $j = 1, \ldots, n$, for a sufficiently large $\gamma > 0$, and assume that $c_n \ge a_n > 0$. By interlacing inequalities, there is $\tilde{C} = \begin{pmatrix} D & y \\ y^t & a \end{pmatrix}$. Show that $C = D + vv^t$ has eigenvalues $c_1 \ge \cdots \ge c_n$.

8. (Extra credit) Suppose $C = A + iB$ so that A and B are Hermitian matrices. Suppose A has eigenvalues $a_1, \ldots, a_n$, and B has eigenvalues $b_1, \ldots, b_n$ such that $a_1^2 \ge \cdots \ge a_n^2$ and $b_1^2 \ge \cdots \ge b_n^2$. If C has singular values $s_1 \ge \cdots \ge s_n$, show that

$$(a_1^2 + b_n^2, \ldots, a_n^2 + b_1^2) \prec (s_1^2, \ldots, s_n^2)$$

and

$$(s_1^2 + s_n^2, \ldots, s_n^2 + s_1^2)/2 \prec (a_1^2 + b_1^2, \ldots, a_n^2 + b_n^2).$$

Hint: $A^2 + B^2 = (CC^* + C^*C)/2$.

9. (Extra credit) Suppose $A = \begin{pmatrix} \tilde{A} & * \\ 0 & * \end{pmatrix}$. Show that

$$s_1(A) \ge s_1(\tilde{A}) \ge s_2(A) \ge s_2(\tilde{A}) \ge \cdots \ge s_{n-1}(\tilde{A}) \ge s_n(A).$$

Hint: Apply interlacing inequalities to A^*A.

10. (Extra credit) Suppose $A, B \in M_n$. For any subsequences $(u_1, \ldots, u_k), (v_1, \ldots, v_k)$ and $(w_1, \ldots, w_k)$ of $(1, \ldots, n)$ such that $w_j = u_j + v_j - j$ for $j = 1, \ldots, k$, and $u_k + v_k - k \le n$, we have

$$\prod_{j=1}^k s_{u_j}(A) s_{v_j}(B) \ge \prod_{j=1}^k s_{w_j}(AB).$$

Hint: By induction on n. Check the case for $n = 2$. Assume that the result holds for matrices of size $n - 1$. If $k = n$, the equality holds. Suppose $k < n$. Let p be the largest integer such that $u_j = j$ for all $j = 1, \ldots, p$, and q be the largest integer such that $v_j = j$ for all $j = 1, \ldots, q$. We may assume that $q \le p$. Let $C = AB$, $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ be orthonormal sets such that

$$B^*Bu_j = s_j(B)^2 u_j \quad \text{and} \quad C^*Cv_j = s_j(C)^2 v_j.$$

Suppose U, V are unitary such that the first $n-1$ columns span a subspace containing $v_1, \ldots, v_1, u_{q+2}, \ldots, u_n$, and $V^*BU = \begin{pmatrix} \tilde{B} & * \\ 0 & * \end{pmatrix}$ with $\tilde{B} \in M_{n-1}$. Let W be unitary such that $W^*BV = \begin{pmatrix} \tilde{A} & * \\ 0 & * \end{pmatrix}$. Then $W^*ABV = \begin{pmatrix} \tilde{A}\tilde{B} & * \\ 0 & * \end{pmatrix}$. Apply induction assumption on $\tilde{A}\tilde{B}$ to finish the proof.