Math 408 Advanced Linear Algebra Homework 7 Sample Solution

Eight points for each question.

1. The sequences (u_1, u_2), (v_1, v_2), (w_1, w_2) satisfy the Horn sequence construction if and only if

   (a) $1 \le u_1 < u_2 \le 3, \quad 1 \le v_1 < v_2 \le 3, \quad 1 \le w_1 < w_2 \le 3;$

   (b) $u_1 + v_1 + u_2 + v_2 = w_1 + w_2 + 3$

   (c) $u_1 + v_1 \ge w_1, \quad u_2 + v_1 \ge w_2, \quad u_1 + v_2 \ge w_2.$

   Then we get the following

Horn sequences

(u_1, v_1)	(u_2, v_2)	(u_3, v_3)
$(1, 2)$	$(1, 2)$	$(1, 2)$
$(1, 2)$	$(1, 3)$	$(1, 3)$
$(1, 2)$	$(2, 3)$	$(2, 3)$
$(1, 3)$	$(1, 2)$	$(1, 3)$
$(1, 3)$	$(1, 3)$	$(2, 3)$
$(2, 3)$	$(1, 2)$	$(2, 3)$

Thompson sequences

(u_1, v_1)	(u_2, v_2)	(u_3, v_3)
$(1, 2)$	$(1, 2)$	$(1, 2)$
$(1, 2)$	$(1, 3)$	$(1, 3)$
$(1, 2)$	$(2, 3)$	$(2, 3)$
$(1, 3)$	$(1, 2)$	$(1, 3)$
$(2, 3)$	$(1, 2)$	$(2, 3)$

2. Let $A, B, C = A+B = (c_{ij}) \in M_n$ be Hermitian with eigenvalues $a_1 \ge \cdots \ge a_n$, $b_1 \ge \cdots \ge b_n$ and $c_1 \ge \cdots \ge c_n$, respectively. By Theorem 3.1.3 and from problem 6 of Homework 5, we know that

$$(c_{11}, \ldots, c_{nn}) \prec (c_1, \ldots, c_n) \prec (a_1 + b_1, \ldots, a_n + b_n).$$

It follows that for all $k \le n$,

$$\sum_{j=1}^{k} c_{jj} \le k \text{ largest diagonal entries of } C \le \sum_{j=1}^{k} (a_j + b_j) \tag{1}$$

Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix}$ and $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{bmatrix}$, where $A_{11}, B_{11} \in M_k$. So $C = \begin{bmatrix} A_{11} + B_{11} & A_{12} + B_{12} \\ A_{12}^* + B_{12}^* & A_{22} + B_{22} \end{bmatrix}$

($\Longleftarrow$) If $A = A_{11} \oplus A_{22}, B = B_{11} \oplus B_{22}$ with $A_{11}, B_{11} \in M_k$ such that A_{11} and B_{11} have eigenvalues $a_1 \ge \cdots \ge a_k, b_1 \ge \cdots \ge b_k$, then we get equality in (1).

($\Longrightarrow$) Conversely, suppose equality is achieved in (1). Consider the set $\mathcal{S}(c_1, \ldots, c_n)$ of all Hermitian matrices with eigenvalues $c_1, \ldots, c_n$. For any $D = [d_{ij}] \in \mathcal{S}$, it holds that

$$\sum_{j=1}^{k} d_{jj} \le \sum_{j=1}^{k} (a_j + b_j) = \sum_{j=1}^{k} c_{jj}$$

because $(d_{11}, \ldots, d_{nn}) \prec (c_1, \ldots, c_n) \prec (a_1 + b_1, \ldots, a_n + b_n)$. From the Claim proven in Problem 2 of Homework 6, we conclude that $A_{12} + B_{12} = 0$ and $A_{11} + B_{11}$ has eigenvalues $c_1, \ldots, c_k$. From the same claim, we know that if $A_{12} \ne 0$ (respectively, if $B_{12} \ne 0$) , then $\mathrm{tr}\,(A_{11}) < a_1 + \cdots + a_k$ (respectively, $\mathrm{tr}\,(B_{11}) < b_1 + \cdots + b_k$). This will lead to a contradiction to the fact that $\mathrm{tr}\,(A_{11} + B_{11}) = c_1 + \cdots + c_k = \sum_{j=1}^{k} a_j + b_j$. Hence $A_{12} = B_{12} = 0$. Therefore, $A = A_{11} \oplus A_{22}, B = B_{11} \oplus B_{22}$ with $A_{11}, B_{11} \in M_k$ such that A_{11} and B_{11} have eigenvalues $a_1 \ge \cdots \ge a_k, b_1 \ge \cdots \ge b_k$.

3. (Weyl's inequalities.)Suppose $A, B, C = A+B \in M_n$ are Hermitian matrices. Let $\{x_1, \ldots, x_n\}$, $\{y_1, \ldots, y_n\}$, $\{z_1, \ldots, z_n\}$ be orthonormal sets such that $Ax_j = \lambda_j(A)x_j, By_j = \lambda_j(B)y_j, Cz_j = \lambda_j(C)z_j$. Suppose i, j are positive integers such that $i + j - 1 \leq n$. Define

$$V_1 = \text{span}\,\{x_i, \cdots, x_n\}, \quad V_2 = \text{span}\,\{y_j, \ldots, y_n\} \quad \text{and} \quad V_3 = \{z_1, \ldots, z_{i+j-1}\}.$$

(a) Note that $\dim V_1 = n - i + 1$, $\dim V_2 = n - j + 1$ and $\dim V_3 = i + j - 1$. Consider the subset $V_2 + V_3$ of $\mathbb{C}^n$. Then $\dim(V_2 + V_3) \leq n$ and from a first course of linear algebra (say, counting the number of nonleading variables of the matrix with columns formed by a basis for V_2 and a basis for V_3), we see that

$$\dim(V_2 + V_3) = \dim V_2 + \dim V_3 - \dim(V_2 \cap V_3) = n + i - \dim(V_2 \cap V_3)$$

Thus $n \geq n + i - \dim(V_2 \cap V_3)$ and hence $\dim(V_2 \cap V_3) \geq i \geq 1$. This means $V_2 \cap V_3$ is a nontrivial subspace. Similarly,

$$\begin{aligned} n \geq \dim(V_1 + (V_2 \cup V_3)) &= \dim V_1 + \dim(V_2 \cup V_3) - \dim(V_1 \cap V_2 \cap V_3) \\ &= n - i + 1 + \dim(V_2 \cup V_3) - \dim(V_1 \cap V_2 \cap V_3) \\ &\geq n + 1 - \dim(V_1 \cap V_2 \cap V_3) \end{aligned}$$

Thus $\dim(V_1 \cap V_2 \cap V_3 \geq 1$. Therefore, $V_1 \cap V_2 \cap V_3$ is not the trivial space and hence there must be unit vector $v \in V_1 \cap V_2 \cap V_3$.

(b) Let v be a unit vector such that $\in V_1 \cap V_2, \cap V_3$. Therefore $v \in V_1$, $v \in V_2$ and $v \in V_3$. First, we know that $A = \sum_{t=1}^{n} \lambda_t(A) x_t x_t^*$. Let $A_X = \sum_{t=i}^{n} \lambda_t(A) x_t x_t^*$. Since $v \in V_1$, then

$$v^* A v = \sum_{t=1}^{n} \lambda_t(A) v^* x_t x_t^* v = \sum_{t=i}^{n} \lambda_t(A) x_t x_t^* = v^* A_X v$$

By Rayleigh's principle $v^* A v = v^* A_X v \leq \lambda_1(A_X) = \lambda_i(A)$.
Similarly, $B = \sum_{t=1}^{n} \lambda_t(B) y_t y_t^*$. Let $B_Y = \sum_{t=j}^{n} \lambda_t(B) y_t y_t^*$. Since $v \in V_2$, then

$$v^* B v = \sum_{t=1}^{n} \lambda_t(B) v^* y_t y_t^* v = \sum_{t=j}^{n} \lambda_t(B) y_t y_t^* = v^* B_Y v$$

By Rayleigh's principle $v^* B v = v^* B_Y v \leq \lambda_1(B_Y) = \lambda_j(B)$.
Lastly, $C = \sum_{t=1}^{n} \lambda_t(C) z_t z_t^*$. Let $C_Z = \sum_{t=1}^{i+j-1} \lambda_t(C) z_t z_t^*$. Since $v \in V_3$, then

$$v^* C v = \sum_{t=1}^{n} \lambda_t(C) v^* z_t z_t^* v = \sum_{t=1}^{i+j-1} \lambda_t(C) z_t z_t^* = v^* C_Z v$$

By Rayleigh's principle $v^* C v = v^* C_Z v \geq \lambda_{i+j-1}(C_Z) = \lambda_{i+j-1}(C)$. Therefore,

$$\lambda_{i+j-1}(C) \leq v^* C v = v^* A v + v^* B v \leq \lambda_i(A) + \lambda_j(B).$$

4. Let $v = (v_1, \ldots, v_n)^t \in \mathbb{C}^n$ and $1 \leq p \leq \infty$. We want to show that $\ell_p(v) \geq \ell_q(v)$. It is enough to show that for any v satisfying $\ell_q(v) = 1$, $\ell_p(v) \geq 1$ by replacing v by $\frac{1}{\ell_q(v)} v$. Note that $\ell_p(v) = \ell_p(v_{abs})$ where $v_{abs} = (|v_1|, \ldots, |v_n|)$. Thus, without loss of generality, we can assume v has nonnegative entries.

Case 1: Suppose $q = \infty$. If $p = q$, then we are done. Hence, assume $p < \infty$. If $\ell_\infty(v) = 1$, then one of the components of v must be 1, say v_1. Then,

$$\ell_p(v) = (1 + v_2 + \cdots + v_n)^{\frac{1}{p}} \geq 1$$

Case 2: Suppose $q < \infty$.

Using Lagrange Multiplier

Let $f(v) = \sum_{j=1}^n v_j^p$ and $g(v) = \left(\sum_{j=1}^n v_j^q\right) - 1$. Let us consider the optimization problem

$$\begin{array}{rl} \min & f(v) \\ \text{s.t.} & g(v) = 1 \end{array}$$

We wish to show that this minimum is ≥ 1. Using Lagrange multipliers, $L(\mu, v) = f(v) - \mu g(v)$.

$$L_{v_j}(\mu, v) = pv_j^{p-1} - \mu q v_j^{q-1} = pv_j^{p-1}\left(1 - \frac{\mu q}{p} v_j^{p-q}\right)$$

If $L_{v_j}(\mu, v) = 0$ then either $v_j = 0$ or $v_j = \left(\frac{p}{\mu q}\right)^{\frac{1}{p-q}}$. Thus, we can assume that the minimum value is attained at a vector of the form $\hat{v} = \left(\frac{p}{\mu q}\right)^{\frac{1}{p-q}} (\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0)$, where $\left(\frac{p}{\mu q}\right)^{\frac{1}{p-q}} = \left(\frac{1}{k}\right)^{\frac{1}{q}}$ since $g(v) = 1$. Note that since $1 \leq p \leq q < \infty$, then $\frac{p}{q} \leq 1$ and hence for any $x \leq 1$, $x^{\frac{p}{q}} \geq x$. Now,

$$f(\hat{v}) = k\left(\frac{1}{k}\right)^{\frac{p}{q}} \geq k \cdot \frac{1}{k} = 1$$

This shows that $\ell_p^p(v) \geq \ell_p^p(\hat{v}) \geq 1$ whenever $\ell_q^q(v) = 1$. This completes the proof when $q < \infty$.

Another way

Suppose $0 \leq b \leq 1$. Note that for any $1 \leq p \leq q < \infty$, $b^p \geq b^q$. To see this, consider the function $h(x) = b^x$. Note that $h'(x) = b^x \ln b \leq 0$ because $b \leq 1$.

Now, going back, the assumption that $\ell_q(v) = 1$ implies that $|v_j| \leq 1$ for all j. Thus $|v_j|^p \geq |v_j|^q$ and thus

$$\ell_p^p(v) = \sum_{j=1}^n |v_j|^p \geq \sum_{j=1}^n |v_j|^q = \ell_q^q(v) = 1 \quad \Longrightarrow \quad \ell_p(v) \geq 1$$

5. Let $p \geq 1$. Suppose $y = (y_k) \in \mathbb{R}^n$ and for some $i \neq j$ $y_i > y_j$ and $d \in (0, y_i - y_j)$. Note that there is $c \in (0,1)$ such that $d = c(y_i - y_j)$. Define $x = (x_k)$ such that

$$x_k = \begin{cases} y_k = (1-c)y_k + cy_k & \text{if } k \notin \{i,j\} \\ y_i - d = (1-c)y_i + cy_j & \text{if } k = i \\ y_j + d = (1-c)y_j + cy_i & \text{if } k = j \end{cases}$$

Then $x = (1-c)y + c\hat{y}$, where $\hat{y}$ is obtained from y by swapping y_i and y_j. Hence $\ell_p(y) = \ell_p(\hat{y})$. Using Minkowski's inequality, we have

$$\ell_p(x) \leq \ell_p((1-c)y) + \ell_p(c\hat{y}) = (1-c)\ell_p(y) + c\ell_p(\hat{y}) = (1-c)\ell_p(y) + c\ell_p(y) = \ell_p(y)$$

Note: This shows that for any $x \prec y$, it holds that $\ell_p(x) \prec \ell_p(y)$ since $x = x_k \prec x_{k-1} \prec \cdots \prec x_0 = y$ where x_t is obtained from x_{t-1} by changing two entries $x_{t-1,i} > x_{t-1,j}$ to $x_{t-1,i} - d_{t-1}$ and $x_{t-1,j} + d_{t-1}$ for some $d \in (0, x_{t-1,i} - x_{t-1,j})$.

6. Let $u = (u_1, \ldots, u_n)^t$ and $v = (v_1, \ldots, v_n)^t$ be such that $(|u_1|, \ldots, |u_n|) \prec_w (|v_1|, \ldots, |v_n|)$. That is,

$$\sum_{i=1}^{k} |u_i| \leq \sum_{i=1}^{k} |v_i| \tag{2}$$

for all $k = 1, \ldots, n$. Let $D = \sum_{i=1}^{n} |v_i| - |u_i| \geq 0$.

If $D = 0$, then $(|u_1|, \ldots, |u_n|) \prec (|v_1|, \ldots, |v_n|)$ and hence we can take $m = 0$ (or $m > 0$, $d = 0$).

Suppose $D > 0$ and suppose WLOG, assume $|u_1| \geq \cdots \geq |u_n|$ and $|v_1| \geq \cdots \geq |v_n|$. If $|u_n| > 0$, take m to be the smallest positive integer such that $\frac{D}{|u_n|} \leq m$ and let $d = \frac{D}{m} \leq |u_n|$. Now, define $\hat{u} = (|u_1|, \ldots, |u_n|, \underbrace{d, \ldots, d}_{m})$ and $\hat{v} = (|v_1|, \ldots, |v_n|, 0, \ldots, 0)$. Note that the entries of $\hat{u}, \hat{v}$ are arranged in nonincreasing order. Then for any $k = 1, \ldots, m$,

$$\sum_{j=1}^{n+k} \hat{u}_j = kd + \sum_{j=1}^{n} |u_j| \leq D + \sum_{j=1}^{n} |u_j| = \sum_{j=1}^{n+k} \hat{v}_j$$

Together with (2), this implies $\hat{u} \prec \hat{v}$.

On the other hand, if $|u_1|, \ldots, |u_r| > 0$ and $|u_{r+1}| = \cdots = |u_n| = 0$ for some $r < n$, then for any $s = r+1, \ldots, n$,

$$\sum_{j=1}^{s} |v_j| - \sum_{j=1}^{r} |u_j| = |v_{r+1}| + \cdots + |v_s| + \sum_{j=1}^{r} |v_j| - \sum_{j=1}^{r} |u_j| > 0$$

since $D > 0$ implies that either $|v_{r+1}| > 0$ or $\sum_{j=1}^{r} |v_j| - \sum_{j=1}^{r} |u_j| > 0$. Take m to be the smallest integer such that for any $s = r+1, \ldots, n$,

$$\frac{sD}{\sum_{j=1}^{s} |v_j| - \sum_{j=1}^{r} |u_j|} \leq m \quad \text{and} \quad \frac{D}{|u_r|} \leq m$$

and $d = \frac{D}{m} \leq |u_r|$. Now, define $\hat{u} = (|u_1|, \ldots, |u_n|, \underbrace{d, \ldots, d}_{m})$ and $\hat{v} = (|v_1|, \ldots, |v_n|, 0, \ldots, 0)$. Note that the entries of $\hat{u}, \hat{v}$ are arranged in nonincreasing order. Then for any $k = 1, \ldots, m$,

$$\sum_{j=1}^{r+k} \hat{u}_j = kd + \sum_{j=1}^{r} |u_j| \leq D + \sum_{j=1}^{r} |u_j| = \sum_{j=1}^{r+k} \hat{v}_j$$

Therefore $\hat{u} \prec \hat{v}$.

From the previous problem that $\ell_p(u) \leq \ell_p(\hat{u}) \leq \ell_p(\hat{v}) = \ell_p(v)$ for any $p \geq 1$.