Math 408 Advanced Linear Algebra Homework 8 Your Name

Eight points for each question.

1. Suppose $\|\cdot\|$ is a matrix/algebra norm on M_n, and suppose $A \in M_n$.

   (a) Let x be an eigenvector of an eigenvalue μ of A, and let $X \in M_n$ be such that every column of X equals x. Show that $AX = \mu X$.

   (b) Deduce from (a) that $r(A) \le \|A\|$.

2. A norm ν on $\mathbb{F}^n$ is compatible with a norm $\|\cdot|$ on M_n if $\nu(Ax) \le \|A\|\nu(x)$ for all $x \in \mathbb{F}^n$. Suppose $\|\cdot\|$ is an algebra norm on M_n. Show that the norm $\nu(x) = \|[x \cdots x]\|$ on $\mathbb{F}^n$ is compatible with $\|\cdot\|$.

3. Show that the following conditions are equivalent for any unitarily invariant norm $\|\cdot\|$ on M_n.

   (a) $\|\cdot\|$ is an algebra norm.

   (b) $\|E_{11}\| \ge 1$.

   (c) $\|A\| \ge s_1(A)$ for any $A \in M_n$.

   [Hint: Note that for any $A, B \in M_n$, $\|A\| \ge \|s_1(A)E_{11}\|$, and $\|AB\| \le \|s_1(A)B\|$.]

   Remark From the above result, one sees that for any unitarily invariant norm $\|\cdot\|$ on M_n, $\xi\|\cdot\|$ is an algebra UI norm if and only if $\xi \ge 1/\|E_{11}\|$.

4. (a) Let $n \ge m \ge 0$. Show that there is $A \in M_n$ such that $\lim_{k\to\infty} A^k$ exists and has rank m.

   (b) Give an example of $B \in M_n$ such that $\lim_{k\to\infty} s_1(B^k) = 1$ but $\lim_{k\to\infty} B^k$ does not exist.

5. Suppose $\|x\|$ is a norm on a complex linear space V satisfying the parallelogram identity

   $$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) \quad \text{for all } x, y \in V.$$

   Define $\langle x, y\rangle = a + ib$ with $2a = \|x+y\|^2 - \|x\|^2 - \|y\|^2, 2b = \|x+iy\|^2 - \|x\|^2 - \|y\|^2$ for any $x, y \in V$.

   (a) Show that $\langle x, y\rangle$ is indeed an inner product.

   (b) Show that $\|x\| = \langle x, x\rangle^{1/2}$.

6. Suppose $A \in M_n$.

   (a) Show that there is a rank one matrix T with $s_1(T) = s_n(A)$ such that $A - T$ is singular.

   (b) Show that if $R \in M_n$ satisfies $s_n(R) > s_1(A)$, then $A - R$ is invertible.

7. Suppose $A \in M_n$ has eigenvalues $\lambda_1, \dots, \lambda_n$ with real parts $a_1 \ge \cdots \ge a_n$. Show that for any $\xi > a_1$ then there is a Hermitian matrix $B \in M_n$ with $\lambda_1(B) < \xi$ such that $A - B$ has all eigenvalues lying in $\{\mu \in \mathbb{C} : \mu + \bar{\mu} < 0\}$.

8. (extra credits) (a) Show that the numerical radius is not an algebra norm.

   (b) Show that the norm defined by $\|A\| = \xi w(A)$ is an algebra norm if and only if $\xi \ge 4$.

9. (extra credits) Suppose $c = (c_1, \ldots, c_n)^t$ with $c_1 \geq \cdots \geq c_n \geq 0$, and $x \in \mathbb{C}^n$. Show that

$$\nu_c(x) = \max\{|c^t Px| : P \in GP_n\} = \sum_{j=1}^{n} c_j Qx$$

for some $Q \in GP_n$ such that Qx has nonnegative entries arranged in descending order.

Recall that GP_n is the set of matrices that can be written as the product of a permutation matrix and a diagonal unitary matrix. Hint: Let $Q \in GP_n$ such that $\nu_c(x) = |c^t Qx|$. Show that we may assume Qx has nonnegative entries. Then show $Qx = (x_1, \ldots, x_n)^t$ satisfies $x_i \geq x_{i+1}$ for any $i = 1, \ldots, n-1$.

10. (extra credits) Recall that for any A$\in M_n$, $W(aI + bA) = \{a + b\mu : \mu \in W(A)\} = a + bW(A)$. Also, $W(A) = W(U^*AU)$ for any unitary $U \in M_n$. Thus, for any $A \in M_2$, we may replace A by $A_0 = U^*(aA + bI)U$ and assume that $A_0 = \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}$ with $a, b \geq 0$. It is known that if $b = 0$ then $W(A_0)$ is a line segment joining $a, -a$; if $a = 0$, then $W(A_0)$ is a circular disk centered at 0 with radius $|b|/2$. Suppose $ab \neq 0$.

    (a) Show that A_0 is unitarily similar to $H + iG$ with $H = \text{diag}\,(\alpha, -\alpha)$ and $G = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$, where $\alpha = \sqrt{a^2 + b^2/4}$ and $\beta = b/2$.

    (b) Show that

    $$W(A_0) = e^{ir}\{v^*Hv + iv^*Gv : v \in \mathbb{C}^n, v^*v = 1\} = e^{ir}\{\alpha x + i\beta y : x + iy \in W(B_0)\},$$

    where $B_0 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ has numerical range equal to $\{\mu : |\mu| \leq 1\}$.

    (c) Deduce from (b) that

    $$W(A_0) = e^{ir}\{x + iy : (x/\alpha)^2 + (y/\beta)^2 \leq 1\}$$

    is an elliptical disk with foci $\pm\sqrt{\alpha^2 - \beta^2} = \pm a$, and semi-minor axis of length $\beta = b/2$.

    Remark The above result is known as the elliptical range theorem asserting that the numerical range of $A \in M_2$ is always an elliptical disk with its eigenvalues μ_1, μ_2 as foci and length of minor axis equal to $\sqrt{(\text{tr}\, AA^*) - |\mu_1|^2 - |\mu_2|^2}$.

11. (extra credits) Suppose $\langle x, y \rangle$ is an inner product on $\mathbb{C}^n$. Let $P = (p_{ij}) \in M_n$ be such that $p_{ij} = \langle e_j, e_i \rangle$ for $1 \leq i, j \leq n$, where $\{e_1, \ldots, e_n\}$ is the standard basis for $\mathbb{C}^n$. Show that $\langle x, y \rangle = y^*Px$ and P is a positive definite matrix.

    Conversely, suppose $Q \in M_n$ is a positive definite matrix. Show that $\langle x, y \rangle = y^*Qx$ is an inner product on $\mathbb{C}^n$.