Math 408 Advanced Linear Algebra Homework 8 Your Name

Eight points for each question.

1. (a) Let $A \in M_n$ and x be an eigenvector corresponding to an eigenvalue μ of A, and let $X \in M_n$ be such that every column of X equals x. Then

$$AX = A\begin{bmatrix} x & \cdots & x \end{bmatrix} = \begin{bmatrix} Ax & \cdots & Ax \end{bmatrix} = \begin{bmatrix} \mu x & \cdots & \mu x \end{bmatrix} = |\mu| \begin{bmatrix} x & \cdots & x \end{bmatrix} = \mu X$$

(b) Suppose $\| \cdot \|$ is a matrix/algebra norm on M_n. Then by definition, $||AB|| \leq ||A|| \cdot ||B||$ for any $B \in M_n$. Thus, if $X \neq 0$ is defined as in (a),

$$||AX|| = ||\mu X|| = |\mu| \cdot ||X|| \leq ||A|| \cdot ||X|| \quad \Longrightarrow \quad |\mu| \leq ||A||$$

Since this holds for any eigenvalue of A, then

$$\max\{|\mu| \mid \mu \text{ is an eigenvalue of } A\} = r(A) \leq ||A||$$

2. Suppose $\| \cdot \|$ is an algebra norm on M_n. Define the norm ν on $\mathbb{F}^n$ by $\nu(x) = \|[x \cdots x]\|$ for all $x \in \mathbb{F}^n$. Then

$$\nu(Ax) = || \begin{bmatrix} Ax & \cdots & Ax \end{bmatrix} || = ||A \begin{bmatrix} x & \cdots & x \end{bmatrix} || \leq ||A|| \cdot || \begin{bmatrix} x & \cdots & x \end{bmatrix} || = ||A|| \nu(x)$$

where the inequality follows from the fact that $|| \cdot ||$ is an algebra norm. Thus ν is compatible with $|| \cdot ||$.

3. Let $|| \cdot ||$ be a unitarily invariant norm on M_n and ν be the symmetric norm on $\mathbb{F}^n$ such that $||A|| = \nu(s(A))$ for all $A \in M_n$.
(a $\Longrightarrow$ b) If $|| \cdot ||$ is an algebra norm, then by problem 1, $||E_{11}|| \geq r(E_{11}) = 1$.
(b $\Longrightarrow$ c) Suppose $||E_{11}|| \geq 1$. Let $y = s(A)$ and $x = (s_1(A), 0, \ldots, 0)$ so that $||s_1(A)|| = \nu(x)$. From theorem 4.4.3,

$$\nu(x) \leq \nu(y) \Longleftrightarrow \nu_k(x) \leq \nu_k(y) \text{ for all } k = 1, \ldots, n$$

Note that $\nu_1(x) = s_1(A) = \nu_1(y)$ and for $k = 2, \ldots, n$, we have

$$\nu_k(x) = s_1(A) \leq s_1(A) + \cdots + s_k(A) = \nu_k(y).$$

Therefore $||A|| = \nu(y) \geq \nu(x) = ||s_1(A)E_{11}|| = s_1(A)||E_{11}|| \geq s_1(A)$.
(c $\Longrightarrow$ a) Suppose $||A|| \geq s_1(A)$ for any $A \in M_n$. Let $x = s(AB)$ and $y = s_1(A)s(B)$ so that

$$||AB|| = \nu(x) \leq \nu(y) = ||s_1(A)B|| \quad \Longleftrightarrow \quad \nu_k(x) \leq \nu_k(y)$$

From extra credit problem #10 in Homework 7, we can deduce that $s_1(A)s_j(B) \geq s_j(AB)$ so that

$$\nu_k(x) = \sum_{j=1}^{k} s_j(AB) \leq \sum_{j=1}^{k} s_1(A)s_j(B) = \nu_k(y)$$

Therefore, for any $A, B \in M_n$ we get $||AB|| \leq ||s_1(A)B|| = s_1(A)||B|| \leq ||A|| \cdot ||B||$. Thus, $|| \cdot ||$ is a matrix norm.
Remark From the above result, one sees that for any unitarily invariant norm $\| \cdot \|$ on M_n, $\xi \| \cdot \|$ is an algebra UI norm if and only if $\xi \geq 1/\|E_{11}\|$.

4. (a) Let $n \geq m \geq 0$. Let U be an $n \times n$ unitary matrix and define $A = U(I_m \oplus 0_{n-m})U^*$ so that $A^k = A$ for any k. Therefore $\lim_{k\to\infty} A^k = A$ and $\text{rank}(A) = m$.

   (b) Let U be an $n \times n$ unitary matrix and $B = U([-1] \oplus 0_{n-1})U^*$ so that $s_1(B) = |e^{i\pi k}| = 1$ and $B^k = U([e^{i\pi k}] \oplus 0_{n-1})U^*$. Clearly, $\lim_{k\to\infty} B^k$ does not exist as $\lim_{k\to\infty} e^{i\pi k}$ does not exist.

5. Suppose $\|x\|$ is a norm on a complex linear space V satisfying the parallelogram identity (PI)

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) \quad \text{for all } x, y \in V.$$

Define $\langle x, y\rangle = a + ib$ with $2a = \|x+y\|^2 - \|x\|^2 - \|y\|^2, 2b = \|x+iy\|^2 - \|x\|^2 - \|y\|^2$ for any $x, y \in V$.

   (a) First, we show that $\langle x_1 + x_2, y\rangle = \langle x_1, y\rangle + \langle x_2, y\rangle$ for any $x_1, x_2, y \in V$. Let

$$\langle x_1 + x_2, y\rangle = a + bi, \quad \langle x_1, y\rangle = c + di \quad \text{and} \quad \langle x_2, y\rangle = e + fi$$

Then

$$\begin{array}{rcl} c+e & = & \frac{1}{2}\left(\|x_1+y\|^2 - \|x_1\|^2 - \|y\|^2\right) + \frac{1}{2}\left(\|x_2+y\|^2 - \|x_2\|^2 - \|y\|^2\right) \\ \text{(rearrangement)} & = & \frac{1}{2}\left(\|x_1+y\|^2 + \|x_2+y\|^2\right) - \frac{1}{2}\left(\|x_1\|^2 + \|x_2\|^2\right) - \|y\|^2 \\ \text{(apply PI 2x)} & = & \frac{1}{4}\left(||x_1+x_2+2y||^2 + ||x_1-x_2||^2\right) - \frac{1}{4}\left(||x_1+x_2||^2 + ||x_1-x_2||^2\right) - ||y||^2 \\ \text{(simplify)} & = & \frac{1}{4}||x_1+x_2+2y||^2 - \frac{1}{4}||x_1+x_2||^2 - ||y||^2 \\ \text{(apply PI)} & = & \frac{1}{4}\Big(2(||x_1+x_2+y||^2 + ||y||^2) - ||x_1+x_2||^2\Big) - \frac{1}{4}||x_1+x_2||^2 - ||y||^2 \\ \text{simplify} & = & \frac{1}{2}(||x_1+x_2+y||^2 - ||x_1+x_2||^2 - ||y||^2) = a \end{array}$$

If we replace y by iy in the above computation, we also get $d+f = b$ so that $\langle x_1+x_2, y\rangle = \langle x_1, y\rangle + \langle x_2, y\rangle$.

Next, we show that $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$ for any $\alpha \in \mathbb{C}$ and $x, y \in V$. Let $\alpha = r + ti$, where $r, t \in \mathbb{R}$. From the previous property we have proven, we get $\langle \alpha x, y\rangle = \langle rx, y\rangle + \langle txi, y\rangle$. Thus, it is enough to show that $\langle ix, y\rangle = i\langle x, y\rangle$ and $\langle cx, y\rangle = c\langle x, y\rangle$ for any real number c. Now, using the parallelogram identity, we get $||x - iy||^2 - ||x||^2 - ||y||^2 = ||x||^2 + ||y||^2 - ||x+iy||^2$ and hence

$$\begin{array}{rcl} \langle ix, y\rangle & = & \frac{1}{2}\left(\|ix+y\|^2 - \|ix\|^2 - \|y\|^2\right) + \frac{i}{2}\left(\|ix+iy\|^2 - \|ix\|^2 - \|y\|^2\right) \\ & = & \frac{1}{2}\left(\|x-iy\|^2 - \|x\|^2 - \|y\|^2\right) + \frac{i}{2}\left(\|x+y\|^2 - \|x\|^2 - \|y\|^2\right) \\ & = & \frac{-1}{2}\left(\|x+iy\|^2 - \|x\|^2 - \|y\|^2\right) + \frac{i}{2}\left(\|x+y\|^2 - \|x\|^2 - \|y\|^2\right) \\ & = & i\Big(\frac{i}{2}\left(\|x+iy\|^2 - \|x\|^2 - \|y\|^2\right) + \frac{1}{2}\left(\|x+y\|^2 - \|x\|^2 - \|y\|^2\right)\Big) \\ & = & i\langle x, y\rangle \end{array}$$

Applying the above property twice, we get

$$\langle -x, y\rangle = \langle i(ix), y\rangle = i\langle ix, y\rangle = i^2\langle x, y\rangle = -\langle x, y\rangle$$

For any positive integer c, we can apply the fact that $\langle x_1 + x_2, y\rangle = \langle x_1, y\rangle + \langle x_2, y\rangle$ repeatedly to get

$$\langle cx, y\rangle = \langle \underbrace{x + \cdots + x}_{c \text{ times}}, y\rangle = \underbrace{\langle x, y\rangle + \cdots + \langle x, y\rangle}_{c \text{ times}} = c\langle x, y\rangle$$

and so $\langle -cx, y\rangle = c\langle -x, y\rangle = -c\langle x, y\rangle$ and $\langle 0, y\rangle = \langle cx - cx, y\rangle = c\langle x, y\rangle - c\langle x, y\rangle = 0$. Thus $\langle cx, y\rangle = c\langle x, y\rangle$ holds true for any integer c. It also holds for any rational number $c = \frac{c_1}{c_2}$ because

$$c_1\langle x, y\rangle = \langle c_1 x, y\rangle = \langle \frac{c_1 c_2}{c_2} x, y\rangle = c_2\langle \frac{c_1}{c_2} x, y\rangle \iff \frac{c_1}{c_2}\langle x, y\rangle = \langle \frac{c_1}{c_2} x, y\rangle$$

Since $f(c) = \langle cx, y\rangle$ is continuous in $\mathbb{R}$ and the set of irrational numbers is dense in $\mathbb{R}$, (meaning, there is a sequence or rational numbers converging to a given irrational number), then it follows that $\langle cx, y\rangle = c\langle x, y\rangle$ for any $c \in \mathbb{R}$ and hence for all $c \in \mathbb{C}$.
Third, we show that $\langle x, y\rangle = \overline{\langle y, x\rangle}$. Note that

$$\begin{aligned} ||x + iy||^2 + ||y + ix||^2 &= ||x + iy||^2 + ||i(x - iy)||^2 \\ &= ||x + iy||^2 + ||x - iy||^2 \\ &= 2(||x||^2 + ||iy||^2) \\ &= 2(||x||^2 + ||y||^2) \end{aligned}$$

From this, we get

$$\begin{aligned} \langle x, y\rangle &= \tfrac{1}{2}\left(\|x + y\|^2 - \|x\|^2 - \|y\|^2\right) + \tfrac{i}{2}\left(\|x + iy\|^2 - \|x\|^2 - \|y\|^2\right) \\ &= \tfrac{1}{2}\left(\|y + x\|^2 - \|y\|^2 - \|x\|^2\right) + \tfrac{i}{2}\left(\|x + iy\|^2 - \|x\|^2 - \|y\|^2\right) \\ &= \tfrac{1}{2}\left(\|y + x\|^2 - \|y\|^2 - \|x\|^2\right) - \tfrac{i}{2}\left(\|y + ix\|^2 - \|x\|^2 - \|y\|^2\right) = \overline{\langle y, x\rangle} \end{aligned}$$

Finally, in (b), we will show that $\langle x, x\rangle = ||x||^2$, from which it will follow that $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$.

(b) The following direct calculation shows $\langle x, x\rangle = ||x||^2$. Note that

$$\begin{aligned} \langle x, x\rangle &= \tfrac{||x+x||^2 - ||x||^2 - ||x||^2}{2} + i\tfrac{||x+ix||^2 - ||x||^2 - ||x||^2}{2} \\ &= \tfrac{||2x||^2 - 2||x||^2}{2} + i\tfrac{||(1+i)x||^2 - 2||x||^2}{2} \\ &= \tfrac{(2\cdot||x||)^2 - 2||x||^2}{2} + i\tfrac{(|1+i|\cdot||x||)^2 - 2||x||^2}{2} \\ &= \tfrac{4\cdot||x||^2 - 2||x||^2}{2} + i\tfrac{2\cdot||x||^2 - 2||x||^2}{2} \\ &= ||x||^2 \end{aligned}$$

Thus, $||x|| = \langle x, x\rangle^{\frac{1}{2}}$

6. Suppose $A \in M_n$.

(a) Let $A \in M_n$ have a singular value decomposition $A = U\text{diag}(s_1(A), \ldots, s_n(A))V$. Let $T = U\text{diag}(0, \ldots, 0, s_n(A))V$ so that $\text{rank}(T) = 1$ and $A - T = U\text{diag}(s_1(A), \ldots, 0)V$ is singular.

(b) Suppose $s_n(R) > s_1(A)$ so that $s_n^2(R) - s_1^2(A) > 0$. Let $0 \neq x \in \mathbb{C}^n$, then

$$\begin{aligned} ||(A - R)x)|| &= ||(R - A)x)|| \\ \text{(triangle inequality)} \quad &\geq ||Rx||^2 - ||Ax||^2 = x^*R^*Rx - x^*A^*Ax \\ \text{(Rayleigh principle)} \quad &\geq s_n^2(R) - s_1^2(A) > 0 \end{aligned}$$

Thus, $Nul(A - R) = \{0\}$ and hence $A - R$ is invertible.

7. Suppose $A \in M_n$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ with real parts $a_1 \geq \cdots \geq a_n$. Let $\xi > a_1$. Then there is a $\delta > 0$ such that $\xi > a_1 + \delta$. Let $B = (a_1 + \delta)I_n$ so that $\lambda_1(B) < \xi$. Let $\mu_j = \lambda_j - (a_1 + \delta)$ for $j = 1, \ldots, n$. Then the eigenvalues of $A - B = A - (a_1 + \delta)I_n$ are $\mu_1, \ldots, \mu_n$ and

$$\mu_j + \bar{\mu}_j = \lambda_j - (a_1 + \delta) + \bar{\lambda}_j - (a_1 + \delta) = 2a_j - 2(a_1 + \delta) = 2(a_j - a_1 - \delta) < 0$$

Therefore, $\mu_j \in \{\mu \in \mathbb{C} : \mu + \bar{\mu} < 0\}$ for all $j = 1, \ldots, n$.