Math 408 Advanced Linear Algebra **Homework 9** **Your Name**

Eight points for each question.

1. Let $A = (a_{ij}) \in M_n$ be such that $|a_{ii}| \geq \sum_{j\neq i} |a_{ij}|$ for every $i = 1, \ldots, n$.

   (a) If $|a_{ii}| > \sum_{j\neq i} |a_{ij}|$ for every $i = 1, \ldots, n$. Show that A is invertible.

   (b) If $a_{ii} \neq 0$ for all i, and $|a_{ii}| > \sum_{j\neq i} |a_{ij}|$ for all but one i's. Show that A is invertible.

   [Hint: Show that there is a diagonal matrix D such that $D^{-1}AD = (b_{ij})$ satisfies $|b_{ii}| > \sum_{j\neq i} |b_{ij}|$ for every $i = 1, \ldots, n$.]

   (c) (Extra 4 points) Show that the conclusion (b) fails if we do not assume that $|a_{ii}| \geq \sum_{j\neq i} |a_{ij}|$ for every $i = 1, \ldots, n$.

2. Let $A = (a_{ij}) \in M_n$ with an eigenvalue λ. Suppose $D = \mathrm{diag}\,(a_{11}, \ldots, a_{nn})$ and $B = A - D$. If $\lambda \neq a_{ii}$ for $i = 1, \ldots, a_{nn}$, show that 1 is an eigenvalue of the matrix $(\lambda I - D)^{-1}B$.

3. Let $A \in M_n$ with columns $v_1, \ldots, v_n$. Show that

   $$|\det(A)| \leq \prod_{j=1}^n \ell_1(v_j) \text{ and } |\det(A)| \leq \prod_{j=1}^n \sum_{k=1}^n |a_{jk}|.$$

   Hint: $\ell_2(v) \leq \ell_1(v)$.

4. Consider $f(z) = z^n + a_1 z^{n-1} + \cdots + a_n$. Suppose μ is a zero of $f(z)$.

   (a) Use the Companion matrix C_f of f to deduce that

   $$|\mu| \leq \max\{1, \sum_{j=1}^n |a_j|\} \text{ and } |\mu| \leq \max(\{1 + |a_j| : 1 \leq j \leq n-1\} \cup \{|a_n|\}).$$

   (b) Show that the Companion matrix of f has singular values $s_2(C_f) = \cdots = s_{n-1}(C_f) = 1$; $s_1(C_f)$ and $s_n(C_f)$ are the singular values of the matrix $\begin{pmatrix} 1 & 0 \\ \gamma & |a_n| \end{pmatrix}$, were $\gamma = \sqrt{\sum_{j=1}^{n-1} |a_j|^2}$.

   (c) Use (b) to conclude that $|\mu| \leq \frac{1}{2}\{\sqrt{(1 + |a_n|)^2 + \gamma^2} + \sqrt{(1 - |a_n|)^2 + \gamma^2}\}$.

5. Let $A \in M_n$ with an eigenvalue μ such that $A - \mu I$ has rank $n - 1$. Suppose x, y are right and left eigenvectors of A corresponding to the eigenvalue μ satisfying $y^*x = 1$. Show that there is an invertible matrix S such that $S^{-1}AS = [\mu] \oplus A_1$ and $A_1 - \mu I_{n-1}$ is invertible.

6. Let $A \in M_n$ be a nonnegative matrix such that $(I + A)^k$ is positive. Show that A has a simple eigenvalue $r(A)$ with positive right and left eigenvectors x and y.

   (Extra eight points) A matrix $A \in M_n$ is irreducible if there is no permutation matrix P such that $P^tAP = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ with $A \in M_k$ with $1 \leq k \leq n - 1$. Show that a nonnegative matrix $A \in M_n$ is such that $(A + I)^{n-1}$ is a positive matrix.

7. Let $V \in M_{4,2}$ satisfy $V^*V = I_2$, and let $P = VV^*$.

   (a) Show that V has a submatrix in M_2 with singular values s_1, s_2 if and only if P has a principal submatrix in M_2 with eigenvalues s_1^2, s_2^2.

   (b) Show that $\det(zI - P) = z^2(z - 1)^2$ so that there is a 2×2 principal submatrix of P with determinant larger than 1/6 and smallest eigenvalue larger than 1/6.

(c) (Extra credit open problem. The solution will earn you an A for the course.) Show that there is a 2×2 principal submatrix of P with smaller eigenvalue larger than or equal to $1/4$.

(d) (Extra credit open problem. The solution will earn you an A for the course, and a research paper.) Prove that for any $n \times k$ matrix V such that $V^*V = I_k$. There is an $k \times k$ principal submatrix of $P = VV^*$ with smallest eigenvalue larger than or equal to $1/n$.