Math 408 Advanced Linear Algebra **Homework 9** **Sample solution**

Eight points for each question.

1. Let $A = (a_{ij}) \in M_n$ be such that $|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|$ for every $i = 1, \ldots, n$.

   (a) For each $i \in \{1, \ldots, n\}$, define $G_i(A) = \left\{ z \in \mathbb{C} \ : \ |z - a_{ii}| \leq \sum_{j \neq i} |a_{ij}| \right\}$. Now, if $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ for every $i = 1, \ldots, n$, then $0 \notin \bigcup\limits_{i=1}^{n} G_i(A)$. By Theorem 5.1.1 (Gershgorin Theorem), all eigenvalues of A are contained in $\bigcup\limits_{i=1}^{n} G_i(A)$. Thus, 0 is not an eigenvalue of A. Therefore, A must be invertible.

   (b) Suppose $a_{ii} \neq 0$ for all i and $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ for all but one i's, without loss of generality say at $i = 1$ (otherwise, apply a permutation similarity). That is

   $$|a_{11}| = \sum_{j \neq 1} |a_{1j}|.$$

   Note that for any $i \in \{2, \ldots, n\}$ such that $a_{i1} \neq 0$, we have

   $$1 < \frac{|a_{ii}| - \sum_{j \neq 1, i} |a_{ij}|}{|a_{i1}|}.$$

   Thus, there must be a $\varepsilon > 1$ such that for all such i,

   $$\frac{1}{a_{11}} \sum_{j \neq 1} |a_{1j}| = 1 < \varepsilon < \frac{|a_{ii}| - \sum_{j \neq 1, i} |a_{ij}|}{|a_{i1}|}.$$

   Then define $B = (b_{ij}) = D^{-1}AD$, where $D = \operatorname{diag}(\varepsilon, 1, \ldots, 1)$. Then

   $$\sum_{j \neq \hat{\imath}} |b_{\hat{\imath}j}| = \frac{1}{\varepsilon} |a_{\hat{\imath}\hat{\imath}}| < |a_{\hat{\imath}\hat{\imath}}| = b_{\hat{\imath}\hat{\imath}}$$

   and for all $i > 1$, we have

   $$\sum_{j \neq i} |b_{ij}| = \left(\sum_{j \neq 1, i} |a_{ij}| \right) + \varepsilon a_{i1} < a_{ii} = b_{ii}$$

   (c) **(Extra 4 points)** The conclusion (b) fails if we do not assume that $|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|$ for every $i = 1, \ldots, n$. A counterexample is $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. Note that $|a_{11}| < |a_{12}|$ and $|a_{22}| > |a_{21}|$ but A is not invertible.

2. Let $A = (a_{ij}) \in M_n$ with an eigenvalue λ and x be an eigenvector of A corresponding to λ. Suppose $D = \operatorname{diag}(a_{11}, \ldots, a_{nn})$ and $B = A - D$. Then $Bx = Ax - Dx = \lambda x - Dx = (\lambda I - D)x$. If $\lambda \neq a_{ii}$ for $i = 1, \ldots, a_{nn}$, then the diagonal matrix $\lambda I - D$ is invertible and hence

   $$(\lambda I - D)^{-1} Bx = (\lambda I - D)^{-1} (\lambda I - D)x = Ix = x.$$

   Therefore x is an eigenvector of $(\lambda I - D)^{-1} B$ corresponding to the eigenvalue 1.

3. Let $A \in M_n$ with columns $v_1, \ldots, v_n$. Let $B = A^*A = (b_{ij})$ and note that

$$\det(B) = \det(A^*A) = \overline{\det(A)}\det(A) = |\det(A)|^2.$$

Since B is positive semidefinite (by Theorem 2.2.5), we can apply HW 5 Problem 3 to conclude that

$$|\det(A)|^2 = \det(B) =\leq \prod_{j=1}^{n} b_{jj} = \prod_{j=1}^{n}\sum_{k=1}^{n} \overline{a_{kj}}a_{jk} = \prod_{j=1}^{n} \ell_2^2(v_j)$$

Taking square roots we get, $|\det(A)| \leq \prod_{j=1}^{n} \ell_2(v_j)$. By HW 7 Problem 4, we have $\ell_2(v_j) \leq \ell_1(v_j)$. Thus $|\det(A)| \leq \prod_{j=1}^{n} \ell_1(v_j)$.

Applying the above result to $A^t = \begin{bmatrix} y_1 & | & \cdots & | & y_n \end{bmatrix}$, we get

$$|\det(A)| = |\det(A^t)| \leq \prod_{j=1}^{n} \ell_1(y_j) = \prod_{j=1}^{n}\sum_{k=1}^{n} |a_{jk}|$$

4. Consider $f(z) = z^n + a_1 z^{n-1} + \cdots + a_n$. Suppose μ is a zero of $f(z)$.

   (a) Note that μ is an eigenvalue of the companion matrix of f, that is

$$C_f = \begin{bmatrix} -a_1 & -a_2 & \cdots & \cdots & -a_n \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} = (c_{ij})$$

   Define the Gershgorin discs G_j centered at the j^{th} diagonal entry of C_f with radius equal to absolute row sum of the j^{th} row. By the Gershgorin theorem, $\mu \in \bigcup_{j=1}^{n} G_j$.
   If $\mu \in G_i$ for $i > 1$, then $|\mu - c_{ii}| = |\mu| \leq \sum_{j \neq i} |c_{ij}| = 1$. On the other hand, if $\mu \in G_1$, then $|\mu - c_{11}| = |\mu + a_1| \leq \sum_{j \neq i} |c_{1j}| = \sum_{k=2}^{n} |a_k|$. Using triangle inequality

$$|\mu| =\leq |\mu + a_1| + |a_1| \leq \sum_{k=1}^{n} |a_k|.$$

   Thus, $|\mu| \leq \max\{1, \sum_{j=1}^{n} |a_j|\}$.
   We can use the same arguments above for C_f^t, which has the same eigenvalues as C_f. Define the Gershgorin discs $\tilde{G}_j$ centered at the j^{th} diagonal entry of C_f with radius equal to absolute column sum of the j^{th} column. By the Gershgorin theorem, $\mu \in \bigcup_{j=1}^{n} \tilde{G}_j$. If $\mu \in \tilde{G}_1$, then $|\mu| \leq |\mu + a_1| + |a_1| \leq 1 + |a_1|$. If $\mu \in G_n$, then $|\mu| \leq |a_n|$. Lastly, if $\mu \in G_j$ for $1 < j < n$, then $|\mu| \leq 1 + |a_j|$. Therefore, $|\mu| \leq \max(\{1 + |a_j| : 1 \leq j \leq n-1\} \cup \{|a_n|\})$.

(b) Let $v^* = \begin{bmatrix}-a_1 & \dots & -a_{n-1}\end{bmatrix}$ so that $C_f = \begin{bmatrix} v^* & -a_n \\ I_{n-1} & 0 \end{bmatrix}$. Then $C_f C_f^* = \begin{bmatrix} v^*v + |a_n|^2 & v^* \\ v & I_{n-1} \end{bmatrix}$.
Note that I_{n-1} is a principal submatrix of $C_f C_f^*$. Using Theorem 3.4.2 (interlacing theorem), we get

$$s_1^2(C_f) \geq \underbrace{\lambda_1(I_{n-1})}_{=1} \geq s_2^2(C_f) \geq \underbrace{\lambda_2(I_{n-1})}_{=1} \geq \cdots \geq s_{n-1}^2(C_f) \geq \underbrace{\lambda_{n-1}(I_{n-1})}_{=1} \geq s_n^2(C_f)$$

Therefore, $s_2(C_f) = \cdots = s_{n-1}(C_f) = 1$.
If $\gamma = ||v|| = \sqrt{\sum_{j=1}^{n-1} |a_j|^2} = 0$. Then it is obvious from the form of $C_f C_f^*$ that $\{|a_n|, 1\} = \{s_1(C_f), s_n(C_f)$. If $||v|| \neq 0$, then let U be an $(n-1)\times(n-1)$ unitary matrix whose last column is $\frac{v}{||v||}$. Consider

$$B = \begin{bmatrix} 0 & I_{n-1} \\ 1 & 0 \end{bmatrix} C_f \begin{bmatrix} U & 0 \\ 0 & sgn(-a_n) \end{bmatrix} = \begin{bmatrix} I_{n-2} & 0 \\ 0 & X \end{bmatrix}, \text{ where } X = \begin{pmatrix} 1 & 0 \\ \gamma & |a_n| \end{pmatrix}.$$

U is unitarily equivalent to C_f, so B has the same singular values as C_f. Therefore $s_1(C_f)$ and $s_n(C_f)$ are the singular values of X.

(c) From (b), we can compute the singular values of X by looking at $XX^* = \begin{bmatrix} 1 & \gamma \\ \gamma & \gamma^2 + |a_n|^2 \end{bmatrix}$ which has characteristic polynomial $p(x) = x^2 - (1 + \gamma^2 + |a_n|^2)x + |a_n|^2$. And therefore

$$s_1^2(C_f) = \frac{1 + \gamma^2 + |a_n|^2 + \sqrt{(1 + \gamma^2 + |a_n|^2)^2 - 4|a_n|^2}}{2}$$

and

$$s_n^2(C_f) = \frac{1 + \gamma^2 + |a_n|^2 + \sqrt{(1 + \gamma^2 + |a_n|^2)^2 - 4|a_n|^2}}{2}$$

Next, consider

$$\begin{aligned} & \left(\tfrac{1}{2}\{\sqrt{(1+|a_n|)^2 + \gamma^2} + \sqrt{(1-|a_n|)^2 + \gamma^2}\}\right)^2 \\ = \; & \tfrac{1}{4}\{2 + 2|a_n|^2 + 2\gamma^2 + 2\sqrt{(1+|a_n|)^2 + \gamma^2)(1-|a_n|)^2 + \gamma^2)}\} \\ = \; & \tfrac{1}{2}\{1 + |a_n|^2 + \gamma^2 + \sqrt{(1-|a_n|^2)^2 + \gamma^4 + 2(1+|a_n|^2)\gamma^2}\} \\ = \; & \tfrac{1+\gamma^2+|a_n|^2+\sqrt{(1+\gamma^2+|a_n|^2)^2 - 4|a_n|^2}}{2} = s_1^2(C_f) \end{aligned}$$

Now, $|\mu| \leq |\lambda_1(C_f)| \leq s_1(C_f)$. Thus, $|\mu| \leq \frac{1}{2}\{\sqrt{(1+|a_n|)^2 + \gamma^2} + \sqrt{(1-|a_n|)^2 + \gamma^2}\}$.

5. Let $A \in M_n$ with an eigenvalue μ such that $A - \mu I$ has rank $n-1$. Then $Nul(A - \mu I)$ and $Nul(A^* - \mu I)$ both have dimension 1. Let x_1, y_1 be a right and left eigenvectors of A with $y^*x = 1$, that is $Nul(A - \mu I) = Span\{x_1\}$ $Nul(A^* - \mu I) = Span\{y_1\}$. Suppose $\{x_2, \dots, x_n\}$ is a basis for $Col(A - \mu I) = Nul(A^* - \mu I)^\perp$. So $y_1^* x_j = 0$ for $j = 2, \dots, n$. Then, $\{x_1, \dots, x_n\}$ must form a basis for $\mathbb{C}^n$. Thus, the matrix $S = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}$ is invertible and S^{-1} has y_1^* as its first row. Denote the rows of S^{-1} by $y_1^*, \dots, y_n^*$. Then define $B = S^{-1}AS = (y_i^* A x_j) = (b_{ij})$

$$b_{i1} = y_i^* A x_1 = \mu y_i^* x_1 = \mu \delta_{i1}$$

and

$$b_{1j} = y_1^* A x_j = \mu y_1^* x_j = \mu \delta_{1j}$$

and thus $B = [\mu] \oplus A_1$, where $rank(A - \mu I) = rank(B - \mu I) = \text{rank}([0] \oplus (A_1 - \mu I)) = n - 1$. This implies $A_1 - \mu I$ is invertible.

6. Let $A \in M_n$ be a nonnegative matrix, then $I + A$ is also nonnegative. Assume that $(I + A)^k$ is positive. Then by Theorem 5.3.1, we conclude that $I + A$ has a simple eigenvalue $r(I + A)$ with positive right and left eigenvectors x and y. However, since the eigenspace of $I + A$ corresponding to an eigenvalue λ is the same as the eigenspace of A corresponding to the eigenvalue $\lambda - 1$. Thus, we conclude that A has a simple eigenvalue $r(A) = r(A+I) - 1$ with positive right and left eigenvectors x and y.

   (Extra eight points) A matrix $A \in M_n$ is irreducible if there is no permutation matrix P such that $P^tAP = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ with $A \in M_k$ with $1 \le k \le n-1$. Show that a nonnegative matrix $A \in M_n$ is irreducible if and only if $(A + I)^{n-1}$ is a positive matrix.

   Proof: Suppose $A \in$ is a nonnegative matrix is reducible. Then up to a permutation similarity, we have $A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ with $A_{11} \in M_k$ with $1 \le k \le n-1$. Then $(I_n + A)^{n-1} = \begin{pmatrix} (I_k + A_{11})^{n-1} & * \\ 0 & (I + A_{22})^{n-k} \end{pmatrix}$, which is not positive.

   To prove the converse, we need to use some graph theory concept. Suppose A is irreducible. We can construct a graph $G(I + A)$ with vertices $1, \ldots, n$ such that there is an arc (i, j) from vertex i to vertex j if $a_{ij} \ne 0$ or $i = j$. It is easy to check that there one can go from vertex i to vertex j by forming a path $i - j_1 - j_2 - j_3 - \cdots - j_k = j$ with k arcs $(i, j_1), (j_1, j_2), \ldots, (j_{k-1}, j_k)$ if and only if $(I + A)^k$ has a nonzero (i, j) entry. Moreover, there is a path from vertex i to vertex j implies that there is a path from vertex i to vertex j in fewer than $n - 1$ steps.

   Consider the set T_1 of the vertices j that can be reached from vertex 1 to j in k steps for $k = 1, 2, \ldots$. We claim that $T_1 = \{1, \ldots, n\}$. Otherwise, relabeling the vertices so that $T_1 = \{1, \ldots, k\}$ with $k < n$. Then permute the rows and columns of A accordingly. We will have $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ with $A_{11} \in M_k$. Note that A_{21} is nonzero. By permutation of the rows and columns with indices $k + 1, \ldots, n$, we may assume that the first row of A_{21} is nonzero. Now, $A^{k-1} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ will be such that $B_{11} \in M_k$ has positive entries so that $A^k = AA^{k-1}$ has a positive entries at the $(k + 1, 1)$ entry so that vertex $k + 1$ can be reached by 1 by a length k path, which is a contradiction. Thus, if A is irreducible, then $T_1 = \{1, \ldots, n\}$, and $(I + A)^{n-1}$ has positive entries.

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7. Let $V \in M_{4,2}$ satisfy $V^*V = I_2$, and let $P = VV^*$.

   (a) V has a 2×2 submatrix W with singular values s_1 and s_2 if and only if W can be written as $W = SV$, where $S = \begin{bmatrix} e_i^* \\ e_j^* \end{bmatrix}$ for some $i, j \in \{1, \ldots, 4\}$ with $i \ne j$ and WW^* has eigenvalues s_1^2 and s_2^2. Note that $WW^* = SVV^*S^*$ is the principal submatrix of P indexed by i and j.

   (b) Note that $P = VV^* \in M_4$ and $V^*V = I_2$ have the same nonzero eigenvalues. Hence the eigenvalues of P must be 1 (with multiplicity 2) and 0 (with multiplicity 2). Thus, $\det(zI - P) = z^2(z-1)^2$. By the interlacing theorem we have that for any 2×2 principal submatrix of P,

$$0 \le s_1^2(P) \le 1 \text{ and } 0 \le s_2^2(P) \le 1$$

Consider the compound matrix $C_2(P)$, which is a 6×6 matrix whose entries are the determinants of all possible 2×2 submatrices of P. Using 5.5.3, we get

$$C_2(P) = C_2(VV^*) = C_2(V)C_2(V^*) = xx^*$$

where $x \in \mathbb{C}^6$ and x^*x is the sum of all the principal minors of P. By Theorem 5.5.2, $x^*x = 1 =$ sum of 2×2 principal minors of I_2. Thus among, the largest principal minor must be greater than or equal to $\frac{1}{6}$. If this principal submatrix has eigenvalues s_1^2 and s_2^2, then $s_1^2 \leq 1$ and thus $\frac{1}{6} \leq s_1^2 s_2^2 \leq s_2^2$.

(c) (**Extra credit open problem**. The solution will earn you an A for the course.) Show that there is a 2×2 principal submatrix of P with smaller eigenvalue larger than or equal to $1/4$.

(d) (**Extra credit open problem**. The solution will earn you an A for the course, and a research paper.) Prove that for any $n \times k$ matrix V such that $V^*V = I_k$. There is an $k \times k$ principal submatrix of $P = VV^*$ with smallest eigenvalue larger than or equal to $1/n$.