Remark 1.5 *For a real matrix $A \in M_n(\mathbb{R})$, we may not be able to find real eigenvalues, and we may not be able to find n linearly independent eigenvectors even if $\det(zI - A) = 0$ has only real roots.*

$-\det(\lambda I - A) = \lambda^2 + 1$

Example 1.6 *Consider $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then A has no real eigenvalues, and B does not have two linearly independent eigenvectors.*

$\hookrightarrow \det(zI - B) = z^2$

$(B - 0I)x = 0$ $\quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\det(\lambda I - A) = \lambda^2 + 1 = 0$, $\lambda = i$ or $-i$

For $\lambda = i$, solve $(A - iI)x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad \cancel{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$

$\left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\quad \therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix}$

For $\lambda = -i$ solve $\left[\begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array}\right] \to \left[\begin{array}{cc|c} i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\quad \therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$.

$\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$A\underbrace{\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}}_{= S} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ $\quad \therefore S^{-1}AS = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

$-i - i = -2i \neq 0$

Remark 1.7 *By the fundamental theorem of algebra, every complex polynomial can be written as a product of linear factors. Consequently, for every $A \in M_n(\mathbb{C})$, the characteristic polynomial has linear factors:*

$$\det(zI - A) = (z - \lambda_1) \cdots (z - \lambda_n).$$

So, we needs only to find n linearly independent eigenvectors.

A

$$\begin{pmatrix} y_1(s) \\ \vdots \\ y_n(s) \end{pmatrix} = S^{-1} \begin{pmatrix} x_1(s) \\ \vdots \\ x_n(s) \end{pmatrix}$$

$y_1(s) = b_{11} x_1(s) + b_{12} x_2(s)$

$y_1'(s) = b_{11} x_1'(s) + b_{12} x_2'(s)$

Math 408 Advanced Linear Algebra Homework 1 Your Name

$(SDS^{-1})(SDS^{-1})$

$\cancel{SAS} = SDS^{-1}$

1. (12 points) Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

   (a) Find the (complex) eigenvalues of A, and the corresponding eigenvectors.

   (b) Determine a complex matrix S such that $S^{-1}AS$ is in diagonal form. $= D$

   (c) Find a formula for A^k for $k = 1, 2, 3, \ldots$.

$\cancel{A = S^{-1}}$ $A = SDS^{-1}$

$A^k = (SDS^{-1})^k = SD^kS^{-1}$

2. (24 points) Suppose $A = SDS^{-1} \in M_n$ such that $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, and suppose S has columns $x_1, \ldots, x_n$ and S^{-1} has rows $y_1^t, \ldots, y_n^t$.

   $D^k = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}^k$

   (a) Show that $A^k = SD^kS^{-1}$.

   (b) Show that $A^k = \sum_{j=1}^{n} \lambda_j^k x_j y_j^t$ for every positive integer k.

   $D^2 = -I$

   (c) If A is invertible, show that $A^k = SD^kS^{-1} = \sum_{j=1}^{n} \lambda_j^k x_j y_j^t$ for every negative integer k.

   $D^3 =$

   (d) For any polynomial $f(z) = a_m z^m + \cdots + a_0$, let $f(A) = a_m A^m + \cdots + a_1 A + a_0 I_n$.

   Show that
   $$f(A) = Sf(D)S^{-1} = \sum_{j=1}^{n} f(\lambda_j) x_j y_j^t.$$

   $A^{-1} = S[D^{-1}]S^{-1}$

   $S^{-1}AS$

   $[S^{-1} A[x_1 \cdots x_n]]$

   (e) Show that $y_j^t x_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$

   (f) Suppose S has rows $u_1^t, \ldots, u_n^t$ and S^{-1} has columns $v_1, \ldots, v_n$. Show that $u_j^t v_j = \delta_{ij}$.

3. (12 points) Let $A \in M_n$ be diagonalizable with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$, and let
   $$f(z) = (z - \lambda_1) \cdots (z - \lambda_k).$$

   (a) Show that A is similar to $\lambda_1 I_{n_1} \oplus \cdots \oplus \lambda_k I_{n_k}$ with $n_1 + \cdots + n_k = n$.

   (b) Show that $f(A) = (A - \lambda_1 I) \cdots (A - \lambda_k I) = 0_n$.

   (c) If $g(z) = \det(zI - A) = (z - \lambda_1)^{n_1} \cdots (z - \lambda_k)^{n_k}$, show that $g(A) = 0_n$.

4. (12 points) Let $x(s) = (x_1(s), \ldots, x_n(s))^t$ be such that $x_1(s), \ldots, x_n(s)$ are differentiable functions. Suppose $x'(s) = (x_1'(s), \ldots, x_n'(s))^t = Ax(s)$, where $\boxed{A = SDS^{-1}}$ with $S =$ diag $(\lambda_1, \ldots, \lambda_n)$, and $y(t) = (y_1(s), \ldots, y_n(s))^t = S^{-1}x(s) = (y_1(s), \ldots, y_n(s))^t$.

   (a) Show that $y'(s) = S^{-1}x'(s)$ so that $\boxed{y'(s) = Dy(s).}$

   (b) Show that $y_i(s) = c_i e^{s\lambda_i}$ with $c_i = y_i(0)$ for $i = 1, \ldots, n$.

   (c) Show that $x(s) = Sy(s) = S(c_1 e^{s\lambda_1}, \ldots, c_n e^{s\lambda_n})^t$.

$\dot{x} = \lambda x \quad x = ke^{\lambda s}$

$x = ke^{\lambda s}$

$S^{-1}S = I_n$

$SS^{-1} = I_n$

$x_1(s)' = \lambda_1 x_1(s) + 0$

$x_2(s)' = 0 + \lambda_2 x_2(s)$

$\begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix}'$

$= \begin{pmatrix} x_1'(s) \\ x_2'(s) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix}$

$A = SDS^{-1}$

$= [x_1 \cdots x_n] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_n \end{bmatrix} \begin{bmatrix} y_1^t \\ \vdots \\ y_n^t \end{bmatrix}$

$= \lambda_1 x_1 y_1^t + \lambda_2 \cdots + \cdots$

1.2 Triangular form and Jordan form

Question What simple form can we get by similarity if $A \in M_n$ is not diagonalizable?

Theorem 1.8 *Let $A \in M_n$ be such that $\det(zI - A) = (z - \lambda_1)\cdots(z - \lambda_n)$. For any permutation $(j_1, \ldots, j_n)$ of $(1, \ldots, n)$, A is similar to a matrix in (upper) triangular form with diagonal entries $\lambda_{j_1}, \ldots, \lambda_{j_n}$.*

$$S^{-1}AS = \begin{bmatrix} \lambda_{i_1} & & * \\ & \ddots & \\ 0 & & \lambda_{i_n} \end{bmatrix}$$

Proof. By induction and block form.

$n = 1$ o.k. $A = [a_{11}]$

Suppose the Theorem holds for matrices of sizes $< n$, $n \geq 2$.

Consider $A \in M_n$.

Let $A x_{i_1} = \lambda_{i_1} x_{i_1}$, $x_{i_1} \neq 0$. and $S_1 = [x_{i_1} | y_2 \cdots | y_n] \in M_n$ be invertible

Apply row reduction to $[x_{i_1} e_1 \cdots e_n]$ and pick up the pivoting column

Then

$$A[x_{i_1} y_2 \cdots y_n]$$
$$= [x_{i_1} y_2 \cdots y_n] \left[\begin{array}{c|c} \lambda_{i_1} & * \\ 0 & \end{array}\right]$$

$A x_{i_1} = \lambda_{i_1} x_{i_1}$

$A y_2 = b_{12} x_{i_1} + b_{22} y_2 + b_{32} y_3 + \cdots + b_{n2} y_n$

$\vdots$

$A y_n = b_{1n} x_{i_1} + b_{n2} y_2 + b_{n3} y_3 + \cdots + b_{nn} y_n$

$$A[x_{i_1} y_2 \cdots y_n] = [x_{i_1} y_2 \cdots y_n] \begin{bmatrix} \lambda_{i_1} & b_{12} & \cdots \\ 0 & b_{22} & \cdots \\ \vdots & \vdots & \\ 0 & b_{n2} & \cdots \end{bmatrix}$$

$\therefore$ $AS_1 = S_1 \begin{bmatrix} \lambda_{i_1} & * \\ 0 & \end{bmatrix}$

$$S_1^{-1} A S_1 = \left[\begin{array}{c|c} \lambda_{i_1} & * \\ \hline 0 & A_1 \end{array}\right]$$

$* = \alpha_1 u_1 + \cdots + \alpha_n u_n$

$= [u_1 \cdots u_n] \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$

So $\exists$ invertible $S_2 \in M_{n-1}$

$$S_2^{-1} A_2 S_2 = \begin{bmatrix} \lambda_{i_2} & & * \\ & \ddots & \\ 0 & & \lambda_{i_n} \end{bmatrix}$$

Let $S = S_1 \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & S_2 \end{array}\right]$. Then $S^{-1} = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & S_2^{-1} \end{array}\right] S_1^{-1}$

and
$$S^{-1} A S = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & S_2^{-1} \end{array}\right] \underline{S_1^{-1} A S_1} \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & S_2 \end{array}\right]$$
$$= \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & S_2^{-1} \end{array}\right] \left[\begin{array}{c|c} \lambda_{i_1} & * \\ \hline 0 & A_1 \end{array}\right] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & S_2 \end{array}\right]$$
$$= \left[\begin{array}{c|c} \lambda_{i_1} & * \\ \hline 0 & S_2^{-1} A_1 \end{array}\right] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & S_2 \end{array}\right]$$
$$= \left[\begin{array}{c|c} \lambda_{i_1} & * \\ \hline 0 & S_2^{-1} A_1 S_2 \end{array}\right] = \left[\begin{array}{c|ccc} \lambda_{i_1} & & * & \\ \hline 0 & \lambda_{i_2} & \ddots & * \\ & 0 & & \lambda_{i_n} \end{array}\right]$$

Done!

Remark. If we consider A^t.

and get
$$S^{-1} A^t S = \begin{bmatrix} \lambda_{i_1} & \ddots & * \\ 0 & & \lambda_{i_n} \end{bmatrix}$$

then
$$\underline{S^t} A \underline{(S^{-1})^t} = \begin{bmatrix} \lambda_{i_1} & \ddots & 0 \\ * & & \lambda_{i_n} \end{bmatrix}$$
$$\parallel$$
$$TAT^{-1}$$

$2\times 2 \quad 3\times 3 \qquad A[X] - X[B] = C \quad 2\times 3 = C$

Lemma 1.9 *Suppose $A \in M_m, B \in M_n$ have no common eigenvalues. Then for any $C \in M_{m,n}$ there is $X \in M_{m,n}$ such that $AX - XB = C$.*

Proof. By the theory of linear equations.

Theorem 1.10 *Suppose $A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \in M_n$ such that $A_{11} \in M_k, A_{22} \in M_{n-k}$ have no common eigenvalue. Then A is similar to $A_{11} \oplus A_{22}$.*

Corollary 1.11 *Suppose $A \in M_n$ has distinct eigenvalues $\lambda_1, \dots, \lambda_k$. Then A is similar to $A_{11} \oplus \cdots \oplus A_{kk}$ such that A_{jj} has (only one distinct) eigenvalue λ_j for $j = 1, \dots, k$.*

Proof of Lemma 1.9

May assume A, B are in triangular form because of the following.

Let ~~$A = S_1^{-1}\hat{A}$~~ $\left.\begin{array}{l} S_1^{-1} A S_1 = \hat{A} \\ S_2^{-1} B S_2 = \hat{B} \end{array}\right\}$ upper triangular ~~triangular~~ lower triangular

To solve $AX - XB = C$

i.e. $S_1^{-1}(S_1 \hat{A} S_1^{-1} X - X S_2 \hat{B} S_2^{-1}) S_2 = S_1^{-1} C S_2$

$$\hat{A} \underbrace{S_1^{-1} X S_2} - \underbrace{S_1^{-1} X S_2} \hat{B} = S_1^{-1} C S_2 = \hat{C}.$$

$\therefore$ We only need to solve

$$\hat{A} Y - Y \hat{B} = \hat{C}, \text{ and then recover } X = S_1 Y S_2^{-1}.$$

Let $\underset{m\times n}{Y = (Y_{ij})}$, $\underset{m\times m}{\hat{A} = (\hat{A}_{ij})}$, $\underset{n\times n}{\hat{B} = (\hat{B}_{ij})}$

Then $n\left\{\begin{bmatrix} \hat{A} & & 0 \\ & \hat{A} & \\ 0 & & \hat{A} \end{bmatrix}\right.$ $\begin{bmatrix} Y_{11} \\ Y_{21} \\ \vdots \\ Y_{m1} \\ Y_{12} \\ \vdots \\ Y_{m2} \\ \vdots \\ Y_{1n} \\ \vdots \end{bmatrix} - \begin{bmatrix} \hat{B}_{11} I_m & \hat{B}_{12} I_m & \cdots \\ \hat{B}_{21} I_m & \ddots & \\ & & \ddots \end{bmatrix} \begin{bmatrix} Y_{11} \\ Y_{21} \\ \vdots \\ Y_{m1} \\ \vdots \\ Y_{mn} \end{bmatrix} = \begin{bmatrix} \hat{C}_{11} \\ \hat{C}_{21} \\ \vdots \\ \hat{C}_{m1} \\ \vdots \\ \hat{C}_{mn} \end{bmatrix}$

$$\underbrace{\left(n\left\{ \begin{bmatrix} \hat{A} & & & & O \\ & \hat{A} & & & \\ & & \hat{A} & & \\ & & & \ddots & \\ O & & & & \hat{A} \end{bmatrix} \right. - \underbrace{\begin{bmatrix} \hat{B}_{11} I_m & & & \\ & \hat{B}_{22} I_m & & \\ & & \ddots & \\ & & & B_{nn} I_m \end{bmatrix}}_{n} \right)} \begin{bmatrix} Y_{11} \\ \vdots \\ Y_{mn} \end{bmatrix} = \begin{bmatrix} \hat{C}_{11} \\ \hat{C}_{21} \\ \vdots \\ \hat{C}_{mn} \end{bmatrix}$$

is in triangular with non-zero diagonal diagonal. $\therefore$ the system is always solvable.