$$A\left\{\left(\begin{bmatrix}\hat{A} & & & 0\\ & \hat{A} & & \\ & & \hat{A} & \\ 0 & & & \ddots \\ & & & & \hat{A}\end{bmatrix} - \underbrace{\begin{bmatrix} B_{11}I_m & & & \\ & B_{22}I_m & & \\ & & \ddots & \\ & & & B_{nn}I_m\end{bmatrix}}_{n}\right)\right.\begin{bmatrix}Y_{11}\\ \vdots \\ \vdots \\ Y_{mn}\end{bmatrix} = \begin{bmatrix}\tilde{C}_{11}\\ \tilde{C}_{21}\\ \vdots\\ \vdots \\ \tilde{C}_{mn}\end{bmatrix}$$

is in triangular with non-zero diagonal diagonal. $\therefore$, the system is always solvable.

Proof of Theorem 1.10

Let $S = \begin{bmatrix} I_k & X \\ 0 & I_{n-k}\end{bmatrix}$ be such that

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22}\end{bmatrix}\begin{bmatrix} I_k & X \\ 0 & I_{n-k}\end{bmatrix} = \begin{bmatrix} I_k & X \\ 0 & I_{n-k}\end{bmatrix}\begin{bmatrix} A_{11} & 0 \\ 0 & A_{22}\end{bmatrix}$$

i.e.,

$$\begin{bmatrix} \underline{A_{11}} & A_{11}X + A_{12} \\ \underline{0} & \underline{A_{22}}\end{bmatrix} = \begin{bmatrix} \underline{A_{11}} & XA_{22} \\ \underline{0} & \underline{A_{22}}\end{bmatrix}$$

~~So~~ i.e., $A_{11}X + A_{12} = XA_{22}$

$$A_{11}X - XA_{22} = -A_{12}$$

Such an X exists by Lemma 1.9. So we get the desired $S = \begin{bmatrix} I_k & X \\ 0 & I_{n-k}\end{bmatrix}$.

2×2 3×3 $A[X] - X[B] = C$ $2\times 3 = C$

Lemma 1.9 *Suppose $A \in M_m, B \in M_n$ have no common eigenvalues. Then for any $C \in M_{m,n}$ there is $X \in M_{m,n}$ such that $AX - XB = C$.*

Proof. By the theory of linear equations.

Theorem 1.10 *Suppose $A = \begin{pmatrix} A_{11} & \boxed{A_{12}} \\ 0 & A_{22} \end{pmatrix} \in M_n$ such that $A_{11} \in M_k, A_{22} \in M_{n-k}$ have no common eigenvalue. Then A is similar to $A_{11} \oplus A_{22}$.*

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Corollary 1.11 *Suppose $A \in M_n$ has distinct eigenvalues $\lambda_1, \dots, \lambda_k$. Then A is similar to $A_{11} \oplus \cdots \oplus A_{kk}$ such that A_{jj} has (only one distinct) eigenvalue λ_j for $j = 1, \dots, k$.*

Proof of Lemma 1.9

May assume A, B are in triangular form because of the following.

Let ~~$A = S_1^{-1}\hat{A}$~~ $S_1^{-1} A S_1 = \hat{A}$ upper triangular; $S_2^{-1} B S_2 = \hat{B}$ ~~triangular~~ lower triangular

To solve $AX - XB = C$.

i.e.. $S_1^{-1}(S_1 \hat{A} S_1^{-1} X - X S_2 \hat{B} S_2^{-1}) S_2 = S_1^{-1} C S_2$

$\hat{A} \underbrace{S_1^{-1} X S_2} - \underbrace{S_1^{-1} X S_2} \hat{B} = S_1^{-1} C S_2 = \hat{C}$

$\therefore$ We only need to solve

$\hat{A} Y - Y \hat{B} = \hat{C}$, and then recover $X = S_1 Y S_2^{-1}$.

Let $\underset{m\times n}{Y = (Y_{ij})}$, $\underset{m\times m}{\hat{A} = (\hat{A}_{ij})}$, $\underset{n\times n}{\hat{B} = (\hat{B}_{ij})}$

Then $n\Big\{ \begin{bmatrix} \hat{A} & & 0 \\ & \hat{A} & \\ 0 & & \ddots \ \hat{A} \end{bmatrix} \begin{bmatrix} Y_{11} \\ Y_{21} \\ \vdots \\ Y_{m1} \\ Y_{12} \\ \vdots \\ Y_{m2} \\ \vdots \\ Y_{1n} \\ \vdots \\ Y_{mn} \end{bmatrix} - \begin{bmatrix} B_{11} I_m & B_{12} I_m & \cdots \\ B_{21} I_m & \ddots & \\ & & \end{bmatrix} \begin{bmatrix} Y_{11} \\ Y_{21} \\ \vdots \\ Y_{m1} \\ \vdots \\ Y_{mn} \end{bmatrix} = \begin{bmatrix} \hat{C}_{11} \\ \hat{C}_{21} \\ \hat{C}_{m1} \\ \vdots \\ \hat{C}_{mn} \end{bmatrix}$

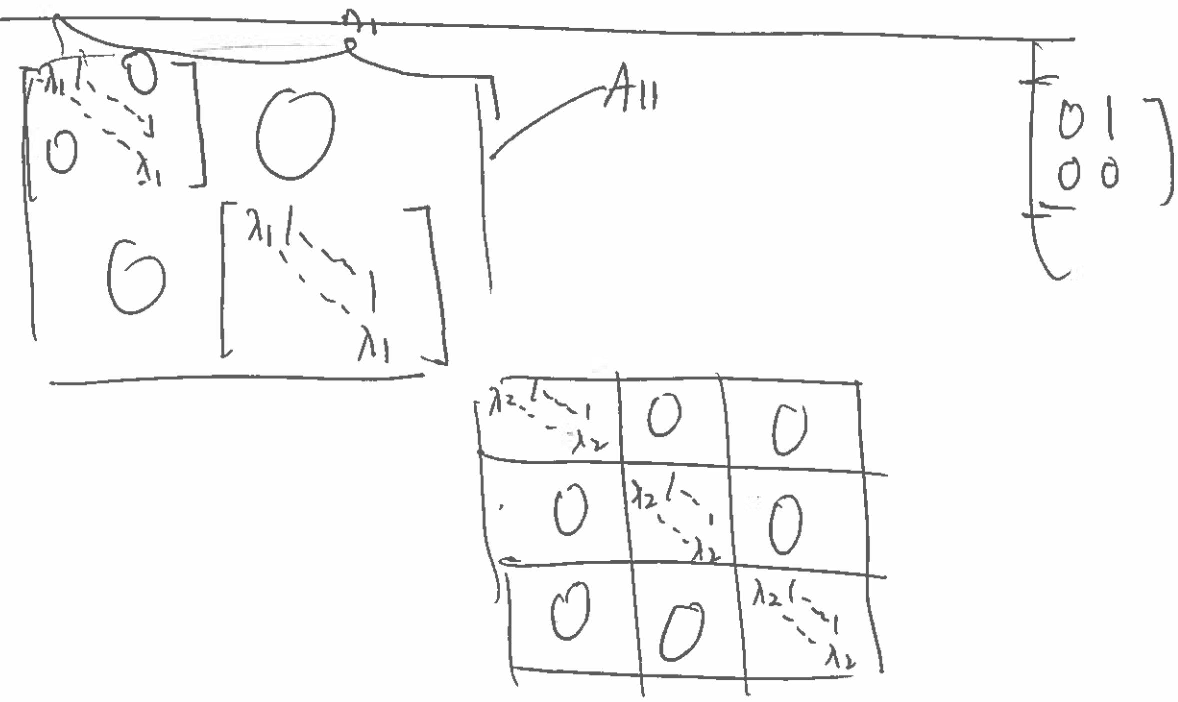
Definition 1.12 *Let $J_k(\lambda) \in M_k$ such that all the diagonal entries equal λ and all super diagonal entries equal 1. Then $J_k(\lambda)$ is call a (an upper triangular) Jordan block of λ of size k.*

Theorem 1.13 *Every $A \in M_n$ is similar to a direct sum of Jordan blocks.*

Proof. We may assume that $A = A_{11} \oplus \cdots \oplus A_{kk}$ ... Then we use a proof of Mark Wildon.

https://www.math.vt.edu/people/renardym/class_home/Jordan.pdf

~~AB~~ $\mathbb{F}$



$$\overset{S^{-1}}{\begin{bmatrix} 1 & & \\ & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}} \begin{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \end{bmatrix} \overset{S}{\begin{bmatrix} 1 & & \\ & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix}} = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

If

$$A_{11} = S_1^{-1}(J_{\lambda_1}(\ ) \oplus J_{\lambda_1}(\ ) \oplus \cdots)\, S_1$$

$$A_{22} = S_2^{-1}(\qquad\qquad)\, S_2$$

— —

— —

Then

$$\begin{pmatrix} S_1 & & 0 \\ & S_2 & \\ 0 & & S_k \end{pmatrix}^{-1} \begin{pmatrix} A_{11} & & 0 \\ & A_{22} & \\ 0 & & A_{kk} \end{pmatrix} \begin{pmatrix} S_1 & & 0 \\ & \ddots & \\ 0 & & S_k \end{pmatrix} = \begin{pmatrix} J_1 & & 0 \\ & J_2 & \\ 0 & & J_m \end{pmatrix}$$

A SHORT PROOF OF THE EXISTENCE OF JORDAN NORMAL FORM

MARK WILDON

Let V be a finite-dimensional complex vector space and let $T : V \to V$ be a linear map. A fundamental theorem in linear algebra asserts that there is a basis of V in which T is represented by a matrix in *Jordan normal form*

$$\begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_k \end{pmatrix}$$

where each J_i is a matrix of the form

$$\begin{pmatrix} \lambda & 1 & \dots & 0 & 0 \\ 0 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & \dots & 0 & \lambda \end{pmatrix} = A_{11}$$

$$S^{-1}A_{11}S = \begin{bmatrix} J_1 & & 0 \\ & J_2 & \\ 0 & & J_k \end{bmatrix}$$

for some $\lambda \in \mathbf{C}$.

$$S^{-1}(A_{11} - \lambda I)S$$
$$\|$$
$$S^{-1}\begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix}S$$
$$\|$$
$$\begin{pmatrix} J_0(k_1) & & * \\ & J_0(k_2) & \\ 0 & & J_0(k_n) \end{pmatrix}$$

We shall assume that the usual reduction to the case where some power of T is the zero map has been made. (See [1, §58] for a characteristically clear account of this step.) After this reduction, it is sufficient to prove the following theorem.

$A : \mathbb{C}^n \to \mathbb{C}^n$

Theorem 1. *If $T : V \to V$ is a linear transformation of a finite-dimensional vector space such that $T^m = 0$ for some $m \geq 1$, then there is a basis of V of the form*

$$u_1, Tu_1, \dots, T^{a_1-1}u_1, \dots, u_k, Tu_k, \dots, T^{a_k-1}u_k$$

where $T^{a_i}u_i = 0$ for $1 \leq i \leq k$.

At this point all the proofs the author has seen (even Halmos' in [1, §57]) become unnecessarily long-winded. In this note we present a simple proof which leads to a straightforward algorithm for finding the required basis.

$$S^{-1}AS = \begin{bmatrix} 0\,1 & 0 \\ 0 & 0\,1 \end{bmatrix} + \lambda I$$

Date: December 2007.

Proof. We work by induction on $\dim V$. For the inductive step we may assume that $\dim V \geq 1$. Clearly $T(V)$ is properly contained in V, since otherwise $T^m(V) = \cdots = T(V) = V$, a contradiction. Moreover, if T is the zero map then the result is trivial. We may therefore assume that $0 \subset T(V) \subset V$. By applying the inductive hypothesis to the map induced by T on $T(V)$ we may find $v_1, \ldots, v_l \in T(V)$ so that

$$v_1, Tv_1, \ldots, T^{b_1-1}v_1, \ldots, v_l, Tv_l, \ldots, T^{b_l-1}v_l$$

is a basis for $T(V)$ and $T^{b_i}v_i = 0$ for $1 \leq i \leq l$.

For $1 \leq i \leq l$ choose $u_i \in V$ such that $Tu_i = v_i$. Clearly $\ker T$ contains the linearly independent vectors $T^{b_1-1}v_1, \ldots, T^{b_l-1}v_l$; extend these to a basis of $\ker T$, by adjoining the vectors $w_1, \ldots, w_m$ say. We claim that

$$u_1, Tu_1, \ldots, T^{b_1}u_1, \ldots, u_l, Tu_l, \ldots, T^{b_l}u_l, w_1, \ldots, w_m$$

is a basis for V. Linear independence may easily be checked by applying T to a given linear relation between the vectors. To show that they span V, we use dimension counting. We know that $\dim \ker T = l + m$ and that $\dim T(V) = b_1 + \ldots + b_l$. Hence, by the rank-nullity theorem,

$$\dim V = (b_1 + 1) + \ldots + (b_l + 1) + m,$$

which is the number of vectors in our claimed basis. We have therefore constructed a basis for V in which T is in Jordan normal form. $\square$

We end by remarking that this proof can be modified to avoid the preliminary reduction. Let λ be an eigenvalue of T. By induction we may find a basis of $(T - \lambda I)V$ in which the map induced by T on $(T - \lambda I)V$ is in Jordan normal form. This basis can then be extended in a similar way to before to obtain a basis for V in which T is in Jordan normal form.

References

[1] P. R. Halmos, *Finite-dimensional vector spaces*, 2nd ed. Undergraduate Texts in Mathematics. Springer, 1987.

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