To find S s.t.

$$S^{-1} \underset{A_{11}}{\begin{bmatrix} \lambda & & * \\ & \ddots & \\ 0 & & \lambda \end{bmatrix}} S = \begin{bmatrix} \lambda & \varepsilon_1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \varepsilon_{n-1} \\ 0 & & & \lambda \end{bmatrix} \qquad \varepsilon_i = 0 \text{ or } 1$$

It suffices to consider.

$$S^{-1} \underbrace{(A_{11} - \lambda I)}_{\hat{A}_{11}} S = \begin{bmatrix} 0 & \varepsilon_1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \varepsilon_{n-1} \\ 0 & & & 0 \end{bmatrix}$$

Then

$$S^{-1} A_{11} S - \underline{S^{-1}(\lambda I) S} = \begin{bmatrix} 0 & \varepsilon_1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \varepsilon_{n-1} \\ 0 & & & 0 \end{bmatrix}$$

$$S^{-1} A_{11} S = \begin{bmatrix} 0 & \varepsilon_1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \varepsilon_{n-1} \\ 0 & & & 0 \end{bmatrix} + \lambda I$$

$$= \begin{bmatrix} \lambda & \varepsilon_1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \varepsilon_{n-1} \\ 0 & & & \lambda \end{bmatrix}$$

$$S^{-1} \hat{A}_{11} S = J_{n_1}(0) \oplus J_{n_2}(0) \oplus \cdots \oplus J_{n_k}(0)$$

$$\hat{A}_{11} [x_1 | \cdots | x_n] = \overset{S}{[x_1 | \cdots | x_n]} \begin{array}{c} \begin{array}{cc} n_1 & n_2 \end{array} \\ \begin{bmatrix} J_{n_1}(0) & 0 & 0 \\ 0 & J_{n_2}(0) & 0 \\ 0 & 0 & \ddots \end{bmatrix} \end{array}$$

$A^{n-1} x_{n_1} \cdots A x_{n_1}, x_{n_1}$ $\quad x_{n_1+1}$ $\quad y_{n_2}$

Want:

$\hat{A}_{11} x_{n_1} = x_{n_1-1}$

$\hat{A}_{11} x_{n_1-1} = x_{n_1-2}$

$x_{n_1}, A x_{n_1}, A^2 x_{n_1}, A^3 x_{n_1}, \ldots, A^{n_1-1} x_{n_1}, A^{n_1} x_{n_1} = 0$

$y_{n_2}, A y_{n_2}, A^2 y_{n_2}, \ldots, A^{n_2-1} y_{n_2}, A^{n_2} y_{n_2} = 0$

A SHORT PROOF OF THE EXISTENCE OF JORDAN NORMAL FORM

MARK WILDON

Let V be a finite-dimensional complex vector space and let $T : V \to V$ be a linear map. A fundamental theorem in linear algebra asserts that there is a basis of V in which T is represented by a matrix in *Jordan normal form*

$$\begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_k \end{pmatrix}$$

$$S^{-1}AS = \left[\begin{array}{c|c|c} A_{11} & 0 & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & 0 & A_{kk} \end{array}\right]$$

where each J_i is a matrix of the form

$$\begin{pmatrix} \lambda & 1 & \dots & 0 & 0 \\ 0 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & \dots & 0 & \lambda \end{pmatrix} = A_1 \qquad 10 \times 10$$

$$S^{-1}A_{11}S = \begin{bmatrix} J_1 & & \\ & J_2 & 0 \\ 0 & & \ddots \; J_k \end{bmatrix}$$

for some $\lambda \in \mathbf{C}$.

We shall assume that the usual reduction to the case where some power of T is the zero map has been made. (See [1, §58] for a characteristically clear account of this step.) After this reduction, it is sufficient to prove the following theorem.

$$S^{-1}(A_{11} - \lambda I)S$$
$$\|$$

$A : \mathbb{C}^n \to \mathbb{C}^n$

Theorem 1. *If $T : V \to V$ is a linear transformation of a finite-dimensional vector space such that $T^m = 0$ for some $m \geq 1$, then there is a basis of V of the form*

$$u_1, Tu_1, \dots, T^{a_1-1}u_1, \dots, u_k, Tu_k, \dots, T^{a_k-1}u_k$$

where $T^{a_i}u_i = 0$ for $1 \leq i \leq k$.

$$S^{-1}\begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix}S$$
$$\|$$
$$\begin{pmatrix} J_0(k_1) & & * \\ & J_0(k_2) & \\ 0 & & \ddots \; J_0(k_m) \end{pmatrix}$$

At this point all the proofs the author has seen (even Halmos' in [1, §57]) become unnecessarily long-winded. In this note we present a simple proof which leads to a straightforward algorithm for finding the required basis.

$$S^{-1}AS = \left[\begin{array}{c|c} {}^{0\,1}_{\;\;0} & 0 \\ \hline 0 & {}^{0\,1}_{\;\;0\,1}{}_{\ddots} \end{array}\right] + \lambda I$$

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Proof. We work by induction on $\dim V$. For the inductive step we may assume that $\dim V \geq 1$. Clearly $T(V)$ is properly contained in V, since otherwise $T^m(V) = \cdots = T(V) = V$, a contradiction. Moreover, if T is the zero map then the result is trivial. We may therefore assume that $0 \subsetneq T(V) \subsetneq V$. By applying the inductive hypothesis to the map induced by T on $T(V)$ we may find $v_1, \ldots, v_l \in T(V)$ so that

$$v_1, Tv_1, \ldots, T^{b_1-1}v_1, \ldots, v_l, Tv_l, \ldots, T^{b_l-1}v_l$$

is a basis for $T(V)$ and $T^{b_i}v_i = 0$ for $1 \leq i \leq l$.

For $1 \leq i \leq l$ choose $u_i \in V$ such that $Tu_i = v_i$. Clearly $\ker T$ contains the linearly independent vectors $T^{b_1-1}v_1, \ldots, T^{b_l-1}v_l$; extend these to a basis of $\ker T$, by adjoining the vectors $w_1, \ldots, w_m$ say. We claim that

$$u_1, Tu_1, \ldots, T^{b_1}u_1, \ldots, u_l, Tu_l, \ldots, T^{b_l}u_l, w_1, \ldots, w_m$$

is a basis for V. Linear independence may easily be checked by applying T to a given linear relation between the vectors. To show that they span V, we use dimension counting. We know that $\dim \ker T = l + m$ and that $\dim T(V) = b_1 + \ldots + b_l$. Hence, by the rank-nullity theorem,

$$\dim V = (b_1 + 1) + \ldots + (b_l + 1) + m,$$

which is the number of vectors in our claimed basis. We have therefore constructed a basis for V in which T is in Jordan normal form. □

We end by remarking that this proof can be modified to avoid the preliminary reduction. Let λ be an eigenvalue of T. By induction we may find a basis of $(T - \lambda I)V$ in which the map induced by T on $(T - \lambda I)V$ is in Jordan normal form. This basis can then be extended in a similar way to before to obtain a basis for V in which T is in Jordan normal form.

References

[1] P. R. Halmos, *Finite-dimensional vector spaces*, 2nd ed. Undergraduate Texts in Mathematics. Springer, 1987.

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$T(V) = 0$

$\begin{bmatrix} 0 \end{bmatrix}$ e_i

V_1

$T(V) = \text{span}\{x_1, \ldots, x_m\}$

$T|: V_1 \to V_1$

V_1 is a lin. transformation on a space V_1 of lower dimension

$T|_{V_1}: V_1 \to V_1$

$T(e_1)=0$

$T(e_2)=0$

Example 1.23 *Let* $T = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. *Then* $Te_1 = 0, Te_2 = 0, Te_3 = e_1 + 3e_2, Te_4 = 2e_1 + 4e_2$.

V_1

So, $T(V) = \boxed{\text{span}\{e_1, e_2\}}$. *Now,* $Te_1 = Te_2 = 0$ *so that* e_1, e_2 *form a Jordan basis for* $T(V)$. *Solving* u_1, u_2 *such that* $T(u_1) = e_1, T(u_2) = e_2$, *we let* $u_1 = -2e_3 + 3e_4/2$ *and* $u_2 = e_3 - e_4/2$. *Thus,* $TS = S(J_2(0) \oplus J_2(0))$ *with*

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \\ 0 & 3/2 & 0 & -1/2 \end{pmatrix}.$$

$\begin{bmatrix} 0&0&1&2 \\ 0&0&3&4 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0&0&1&2 \\ 0&0&3&4 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{cc|cc} 0 & 1 & 0 & \\ 0 & 0 & & \\ \hline 0 & & 0 & 1 \\ & & 0 & 0 \end{array}\right]$

$T|_{V_1}: V_1 \to V_1$

Example 1.24 *Let* $T = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$. *Then* $Te_1 = 0, Te_2 = e_1, Te_3 = 2e_1 + e_2$. *So,* $T(V) =$ $\text{span}\{e_1, e_2\}$, *and* $e_2, Te_2 = e_1$ *form a Jordan basis for* $T(V)$. *Solving* u_1 *such that* $T(u_1) = e_2$, *we have* $u_1 = (-2e_2 + e_3)/3$. *Thus,* $TS = SJ_3(0)$ *with*

$e_2, T(e_2) = e_1$

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1/3 \end{pmatrix}.$$

$T(u) = e_2$

$\begin{bmatrix} 0&1&0 \\ 0&0&1 \\ 0&0&0 \end{bmatrix}$

$\left(\begin{array}{ccc|c} 0&1&2&0 \\ 0&0&3&1 \\ 0&0&0&0 \end{array}\right)$

Remark 1.14 *The Jordan form of A can be determined by the rank/nullity of $(A-\lambda_j I)^k$ for $k = 1, 2, \ldots$. So, the Jordan blocks are uniquely determined.*

Example 1.15 *Suppose $A \in M_9$ has distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3$ such that $A - \lambda_1 I$ has rank 8, $A - \lambda_2 I$ has rank 7, $(A-\lambda_2 I)^2$ and $(A-\lambda_2 I)^3$ have rank 5, $A - \lambda_3 I$ has rank 6, $(A-\lambda_3 I)^2$ and $(A-\lambda_3 I)^3$ have rank 5. Then the Jordan form of A is*

$$J_1(\lambda_1) \oplus \boxed{J_2(\lambda_2)} \oplus \boxed{J_2(\lambda_2)} \oplus \boxed{J_1(\lambda_3)} \oplus \boxed{J_1(\lambda_3)} \oplus \boxed{J_2(\lambda_3)}.$$

Assume

$$A \sim \begin{bmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{bmatrix} \sim S \begin{bmatrix} J_{n_1}(0) & & 0 \\ & \ddots & \\ 0 & & J_{n_k}(0) \end{bmatrix} S^{-1} \quad n$$

$A^m = 0$

	A	A^2	A^3	
rank	$n-k$			
nullity	$k = \ell_1$	ℓ_2	ℓ_3	$\ell_m = n$

$\ell_2 - \ell_1 = \#$ of Jordan block > 1.

$\ell_3 - \ell_2 = \#$ of Jordan block > 2

Example 1.15. $(A-\lambda_1 I)$ has rank 8 $\Rightarrow$ only one eigenvector corr. to λ_1. only one Jordan block

Free by later analysis $(*)$ $\rightarrow$ Check $(A-\lambda_1 I)^2$ has rank 8 $\Rightarrow$ only one (size)? Jordan block $\therefore J_1(\lambda_1)$

λ_2:
- $(A-\lambda_2 I)$ has rank 7. $\Rightarrow$ 2 Jordan blocks.
- $(A-\lambda_2 I)^2$ has rank 5 $\Rightarrow$ 2 Jordan blocks of sizes ≥ 2
- $(A-\lambda_2 I)^3$ has rank 5 $\Rightarrow$ no Jordan block of size ≥ 3

λ_3:
- $(A-\lambda_3 I)$ has rank 6 $\Rightarrow$ 3 Jordan blocks
- $(A-\lambda_3 I)^2$ has rank 5 $\Rightarrow$ 1 block size ≥ 2
- $(A-\lambda_3 I)^3$ has rank 5 $\Rightarrow$ no block size ≥ 3

1.3 Implication of Jordan form

Theorem 1.16 *Two matrices are similar if and only if they have the same Jordan form.*

Remark 1.17 *If $A = S(J_1 \oplus \cdots \oplus J_k)S^{-1}$, then $A^m = S(J_1^m \oplus \cdots \oplus J_k^m)S^{-1}$.*

Proof ~~$A \sim B$~~ ~~Then~~ $A = S(J_1 \oplus \cdots \oplus J_\ell)S^{-1}$

$B = T(J_1 \oplus \cdots \oplus J_\ell)T^{-1}$

$\Rightarrow$ $S^{-1}AS = T^{-1}BT$

$\Rightarrow$ $(TS^{-1})A(ST^{-1}) = B$

Conversely if $A = S^{-1}BS$

then ~~$A \to \lambda$~~ A, B have the same e.v. & $\operatorname{rank}(A-\lambda_i I)^\ell = \operatorname{rank}(B-\lambda_i I)^\ell$

$\forall$ e.v. λ_i & $\ell \in \mathbb{N}$

$\therefore$ A & B have the same Jordan form. $\square$

Remark

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ are not similar.

$S^{-1}AS = B \Rightarrow A = SBS^{-1} = 0$

Theorem 1.18 *Let $J_k(\lambda) = \lambda I_k + N_k$, where $N_k = \sum_{j=1}^{k-1} E_{j,j+1}$. Then $J_k(\lambda)^m = \sum_{j=0}^{m} \lambda^{m-j} N_k^j \binom{m}{k}$ where $N_k^0 = I_k$, $N_k^j = 0$ for $j \geq k$, and N_k^j has one's at the jth super diagonal (entries with indexes $(\ell, \ell + j)$) and zeros elsewhere.*

Note:

$$A = S^{-1}(J_1 \oplus \cdots \oplus J_\ell)S$$

$$A^m = S^{-1}(J_1^m \oplus \cdots \oplus J_\ell^m)S$$

$$\begin{bmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_\ell \end{bmatrix}^m$$

$$J_k(\lambda)^m = \left(\lambda I + \underbrace{\begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}}_{N_k}\right)^m$$

$$= \lambda^m I + \binom{m}{1}\lambda^{m-1} N_k + \binom{m}{2}\lambda^{m-2} N_k^2 + \cdots + \binom{m}{k}\lambda^{m-k} N_k^k$$

$N_k^k \to 0$

$$N_k = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$$

$$N_k^2 = \begin{bmatrix} 0 & 0 & 1 & & 0 \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & \ddots & 0 \\ & & & & 0 \end{bmatrix}$$

$$N_k^3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ & \ddots & \ddots & \ddots & 1 \\ & & \ddots & \ddots & 0 \\ 0 & & & \ddots & 0 \\ & & & & 0 \end{bmatrix}$$

Example.

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ & 2 & 1 & 0 \\ & & 2 & 1 \\ & & & 2 \end{bmatrix}^{10} = 2^{10} I + \binom{10}{1} 2^9 \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}$$

$$+ \binom{10}{2} 2^8 \begin{bmatrix} 0 & 0 & 1 & 0 \\ & & \ddots & 1 \\ & & 0 & 0 \\ & & & 0 \end{bmatrix} + \binom{10}{3} 2^7 \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$