1

Algorithms for computing Jordan form

$$S^{-1}AS = \begin{bmatrix} \lambda_1 \ \lambda_1 \ldots & * & * \\ 0 & \ddots & * \\ 0 & & \lambda_k \end{bmatrix}$$

2

$$S_2^{-1}(S^{-1}AS)S_2 = \begin{bmatrix} \lambda_1 \ * \\ 0 \ \lambda & 0 & 0 \\ & \lambda_2 \ldots * \\ & \quad \lambda_2 & 0 \\ & & \lambda \ \ldots \ \lambda_k \end{bmatrix}$$

3

$$T = A_{ii} - \lambda_i I = \begin{bmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{bmatrix} n_i$$

n_i

$$\begin{pmatrix} 3 & * & * \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = T$$

$T(\mathbb{C}^3) = \text{Span}\{e_1\} = V_1$

$T|_{V_1} : V_1 \to V_1$

$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to u_1$$

$T(e_1) = 0$

Find $u_1 \in \mathbb{C}^3$ s.t. $T(u_1) = e_1$.

$S = \left[\boxed{u_1, \ T(u_1) = e_1, \ u_1} \ , \ w_1 \right]$ — $\ker(T)$

$w_1 \neq \lambda e_1$

$\ker(T)$

$$TS = T[T(u_1)\, u_1 \,|\, w_1] = [T(u_1)\, u_1\, w_1] \begin{pmatrix} 0 & 1 & \\ 0 & 0 & \\ 0 & 0 & 1 \end{pmatrix}$$

$A \in M_3$

$\det(zI - A) = z^3$

$$A \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \left[\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right], \quad \begin{bmatrix} 0 & & \\ & 0 & \\ & & 0 \end{bmatrix}$$

$\det(zI - A) = (z-1)^2 (z-2)^2$

$$A \sim \left[\begin{array}{cc|cc} 1 & & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 2 & \\ 0 & 0 & 0 & 2 \end{array}\right]$$

$A \in M_5$ $\quad 1, \ i$

1° A $\quad 1, i, i, i, i$ $\quad \sim \left[\begin{array}{c|cccc} 1 & & 0 & & \\ \hline 0 & i & & & \\ & & i & & \\ & & & i & \\ & & & & i \end{array}\right]$

2° A $\quad 1, 1, i, i, i$ $\quad \sim \left[\begin{array}{cc|ccc} 1 & & & 0 & \\ & 1 & & & \\ \hline & & i & & \\ & 0 & & i & \\ & & & & i \end{array}\right]$

4° $\quad \sim \left[\begin{array}{cccc|c} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ \hline & & & & i \end{array}\right]$

$$\begin{bmatrix} 3 & 1 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \sim \left[\begin{array}{cc|c} 3 & 1 & 0 \\ 0 & 3 & 0 \\ \hline 0 & 0 & 3 \end{array}\right]$$

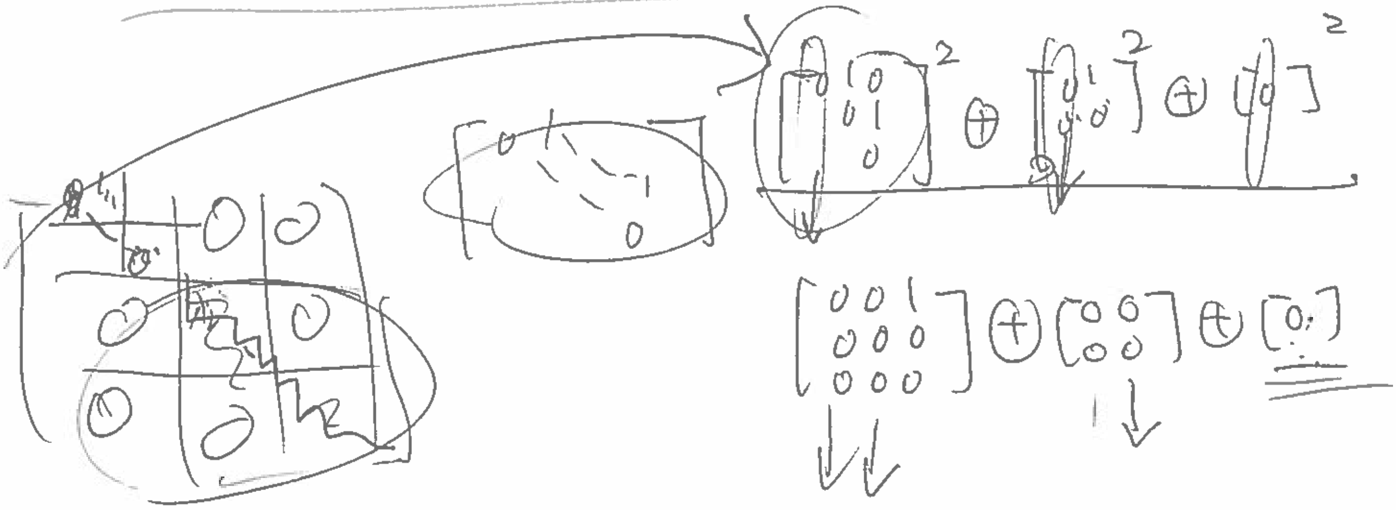
$\mathrm{Ker}(A-3I)$ · has dim 2.

$\mathrm{Ker}(A-3$

$A \in M_n$

a-$\mathrm{Ker}(A-\lambda I) = \ell_1$

$\mathrm{Ker}(\quad)^2 = \ell_2 \qquad \ell_2 - \ell_1$



1.3 Implication of Jordan form

Theorem 1.16 *Two matrices are similar if and only if they have the same Jordan form.*

Remark 1.17 *If* $A = S(J_1 \oplus \cdots \oplus J_k)S^{-1}$, *then* $A^m = S(J_1^m \oplus \cdots \oplus J_k^m)S^{-1}$.

Proof $\cancel{A \sim B}$ ~~then~~ $A = S(J_1 \oplus \cdots \oplus J_\ell)S^{-1}$

$$B = T(J_1 \oplus \cdots \oplus J_\ell)T^{-1}$$

$\Rightarrow$ $S^{-1}AS = T^{-1}BT$

$\Rightarrow$ $(TS^{-1})A(ST^{-1}) = B$

Conversely if $A = S^{-1}BS$

then A, B have the same e.v. & $\operatorname{rank}(A-\lambda_i I)^\ell = \operatorname{rank}(B-\lambda_i I)^\ell$

$\forall$ e.v. λ_i & $\ell \in \mathbb{N}$.

$\therefore$ A & B have the same Jordan form. $\square$

Remark

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ are not similar.

$$\boxed{S^{-1}AS = B \Rightarrow A = SBS^{-1} = 0}$$

$$J_k(\lambda) = \begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix} = \lambda I + \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}$$

Theorem 1.18 *Let $J_k(\lambda) = \lambda I_k + N_k$, where $N_k = \sum_{j=1}^{k-1} E_{j,j+1}$. Then $J_k(\lambda)^m = \sum_{j=0}^{m} \lambda^{m-j} N_k^j \binom{m}{k}$ where $N_k^0 = I_k$, $N_k^j = 0$ for $j \geq k$, and N_k^j has one's at the jth super diagonal (entries with indexes $(\ell, \ell+j)$) and zeros elsewhere.*

Note:

$$A = S^{-1}(J_1 \oplus \cdots \oplus J_\ell) S$$

$$A^m = S^{-1}(J_1^m \oplus \cdots \oplus J_\ell^m) S$$

$$\begin{bmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_\ell \end{bmatrix}^m$$

$$J_k(\lambda)^m = \left(\lambda I + \underbrace{\begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}}_{N_k}\right)^m$$

$$= \lambda^m I + \binom{m}{1}\lambda^{m-1} N_k + \binom{m}{2}\lambda^{m-2} N_k^2 + \cdots + \boxed{\binom{m}{k} N_k^k}$$

$N_k^k \to 0$

$$N_k = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$$

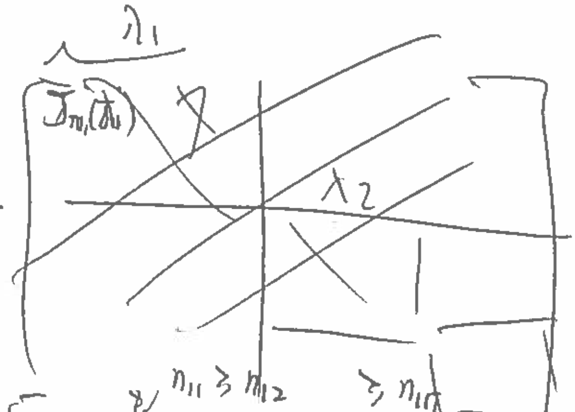
$$N_k^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ & \ddots & \ddots & \ddots \\ & & \ddots & 0 \\ & & & 0 \end{bmatrix}$$

$$N_k^3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ & \ddots & \ddots & \ddots & \ddots \\ & & \ddots & \ddots & 0 \\ 0 & & & & 0 \end{bmatrix}$$

Example:

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ & 2 & 1 & 0 \\ & & 2 & 1 \\ 0 & & & 2 \end{bmatrix}^{10} = 2^{10} I + \binom{10}{1} 2^9 \underbrace{\begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}}_{N_k}$$

$$+ \binom{10}{2} 2^8 \begin{bmatrix} 0 & 0 & 1 & 0 \\ & 0 & 0 & 1 \\ & & 0 & 0 \\ & & & 0 \end{bmatrix} + \binom{10}{3} 2^7 \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Let $A = S$  S^{-1}

$n_{11} \geq n_{12} \geq \cdots \geq n_{1r_1}$

$$= S \begin{bmatrix} J_{n_{11}}(\lambda_1) \oplus J_{n_{12}}(\lambda_1) \oplus \cdots \oplus J_{n_{1r_1}}(\lambda_1) \\ \oplus J_{n_{21}}(\lambda_2) \oplus J_{n_{22}}(\lambda_2) \oplus \cdots \oplus J_{n_{2r_2}}(\lambda_2) \\ \vdots \\ \oplus J_{n_{k1}}(\lambda_k) \oplus \cdots \oplus J_{n_{kr_k}}(\lambda_k) \end{bmatrix} S^{-1}$$

$n_{k1} \geq \cdots \geq n_{kr_k}$

Then. $g(z) = (z-\lambda_1)^{n_{11}} (z-\lambda_2)^{n_{21}} \cdots (z-\lambda_k)^{n_{k1}}$

where n_{j1} is the size of the largest Jordan block corresponding to the eigenvalue λ_j

For every polynomial function $f(z) = a_k z^k + \cdots + a_0$, let

$$f(A) = a_k A^k + \cdots + a_0 I_n \quad \text{for } A \in M_n.$$

Theorem 1.19 (Cayley-Hamilton) *Let $A \in M_n$ and $f(z) = \det(zI - A)$. Then $f(A) = 0$.*

$$f(z) = \det(zI - A) = (z-\lambda_1)\cdots(z-\lambda_n)$$

$$= (z-\mu_1)^{n_1} \cdots\cdots (z-\mu_k)^{n_k}$$

$\mu_1 \cdots \mu_k$ are the distinct eigenvalues

~~f(AI-A~~

$\det(AI-A) = 0 \in \mathbb{C}$ X

$\det(zI-A)$

$= (z-\lambda_1)\cdot(z-\lambda_n)$

$= z^n + a_{n-1}z^{n-1} + \cdots + a_0$

$A^n + a_{n-1}A^{n-1} + \cdots + a_0 I_n$

$= 0_n \in M_n$

$$f(A) = (A-\mu_1 I)^{n_1} \cdots\cdots (A-\mu_k I)^{n_k} \in M_n$$

$$S^{-1}AS = J_1 \oplus \cdots \oplus J_\ell, \quad A = S(J_1 \oplus \cdots \oplus J_\ell)S^{-1}$$

$$f(A) = S f(J_1 \oplus \cdots \oplus J_\ell) S^{-1}$$

$$= S f\begin{bmatrix} J_1 & 0 \\ 0 & J_\ell \end{bmatrix} S^{-1} = S \begin{bmatrix} f(J_1) & & 0 \\ & \ddots & \\ 0 & & f(J_\ell) \end{bmatrix} S^{-1}.$$

~~We will~~ show $f(J_j) = 0$

Let $J_j = \begin{bmatrix} \mu_r & 1 & 0 \\ & \ddots & 1 \\ 0 & & \mu_r \end{bmatrix}$.

$$f(J_j) = (J_j - \mu_1 I)^{n_1} \cdots\cdots (J_j - \mu_k)^{n_k}$$

$(J_j - \mu_j I)^{n_j}$

$\begin{bmatrix} \mu_j & & * \\ & \ddots & \\ 0 & & \mu_j \end{bmatrix}$

$q\left\{\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ & \ddots & 1 \\ & & 0 \end{bmatrix}}_{q}\right.^{n_j} = 0$ because $q \le n_j$.

BPI

$$A = S \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 2 & & & 0 & \\ & & & 2 & & & \\ & 0 & & & 3 & & \\ & & & & & 3 & \\ & & & & & & 3 \end{bmatrix} S^{-1}$$

If $f(z) = \det(zI - A) = (z-1)^2 (z-2)^2 (z-3)^3$ — degree 7.

Then

$$f(A) = 0$$

$$h(z) = (z-1)(z-2)(z-3) \quad \text{— degree 3.}$$

$$h(A) = (A - I)(A - 2I)(A - 3I)$$

$$= S(D - I)(D - 2I)(D - 3I) S^{-1}$$

$$A = S \left[\begin{array}{cc|cc|ccc} 1 & 1 & & & & & \\ 0 & 1 & & & & & \\ \hline & & 2 & 0 & & & \\ & & 0 & 2 & & & \\ \hline & & & & 3 & 1 & 0 \\ & & & & 0 & 3 & 0 \\ & & & & 0 & 0 & 3 \end{array}\right] S^{-1}$$

$(A_1 - I)$ $(A_3 - 3I)^2$

Let $f(z) = \det(zI - A)$ Then $f(A) = O_7 \in M_7$

Let $\hat{h}(z) = (z-1)^2 (z-2)(z-3)^2$.

Then $h(A) = S \left(\hat{h}(J) \right) S^{-1}$

$$= S \begin{pmatrix} \hat{h}(J_1) & 0 & 0 \\ 0 & \hat{h}(J_2) & 0 \\ 0 & 0 & \hat{h}(J_3) \end{pmatrix} S^{-1} \neq 0$$

$$(J_1 - I)^2 (J_1 - 2I)(J_1 - 3I)$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}$$

$$g(z) = (z - \lambda_1)^{r_1} (z - \lambda_2)^{r_2} \cdots (z - \lambda_k)^{r_k}$$

$r_1 \geq n_{11}$ $\geq n_{k\cdots}$

Definition 1.20 *Let $A \in M_n$. Then there is a unique monic polynomial*

$$\boxed{m_A(z) = x^m + a_1 x^{m-1} + \cdots + a_m}$$

such that $m_A(A) = 0$. It is the minimal polynomial of A.

Theorem 1.21 *A polynomial $f(z)$ satisfies $f(A) = 0$ if and only if it is a multiple of the minimal polynomial of A.*

$O_n g(A) + r(A) = r(A)$

$\Rightarrow f(A) = m_A(A) g(A) + r(A)$

Remark 1.22 *By the Euclidean algorithm, $\boxed{f(z)} = \boxed{m_A(z)q(z) + r(z)}$ so that $r(z)$ has degree at most $m-1$ if $m_A(z)$ has degree m, and $f(A) = r(A)$. Now, the Jordan form of $f(A) = r(A)$ can be determine in terms of the Jordan form of A and $r(z)$.*

Remark: Suppose $A \in M_n$ has distinct eigenvalues $\lambda_1, \dots, \lambda_k$. and the ~~Ford~~ max size Jordan block of λ_j is r_j. $j = 1, \dots, k$. Then $m_A = (z-\lambda_1)^{r_1} \cdots (z-\lambda_k)^{r_k}$.

Note: $k \leq \deg m_A(z) \leq n$.

Note: Suppose $A^m + a_{m-1}A^{m-1} + \cdots + a_0 I = O_n \in M_n$ & $a_0 \neq 0$.

Then
$$a_0 I_n = -(A^m + a_{m-1}A^{m-1} + \cdots + a_1 A)$$
$$= -(A^{m-1} + a_{m-1}A^{m-2} + \cdots + a_1 I)A$$

so that
$$I_n = \underbrace{-\frac{1}{a_0}(A^{m-1} + \cdots + a_1 I)}_{A^{-1}} A$$

Example: $A^3 + 2A^2 + 2I = 0$

$$2I = -A^3 - 2A^2 = -(A^2 + 2A)A$$

$$\therefore A^{-1} = -(A^2 + 2A)/2.$$