$\lambda_i \in \mathbb{R}$.

1.4 Final remarks

$\det(zI - A) = \boxed{(z-\lambda_1) \cdots (z-\lambda_n)}$.

- If $A \in M_n(\mathbb{R})$ has only real eigenvalues, then one can find a real invertible matrix such that $S^{-1}AS$ is in Jordan form.
- If $A \in M_n(\mathbb{R})$, then there is a real invertible matrix such that $S^{-1}AS$ is a direct sum of real Jordan blocks, and $2k \times 2k$ generalized Jordan blocks of the form $(C_{ij})_{1\le i,j\le k}$ with $C_{11} = \cdots = C_{kk} = \begin{pmatrix} \mu_1 & \mu_2 \\ -\mu_2 & \mu_1 \end{pmatrix}$; $C_{12} = \cdots = C_{k-1,k} = I_2$, and all other blocks equal to 0_2.
- The proof can be done by the following two steps.

  First of all find the Jordan form of A. Then group $J_k(\lambda)$ and $J_k(\bar\lambda)$ together for any complex eigenvalues, and find a complex S such that $S^{-1}AS$ is a direct sum of the above form.

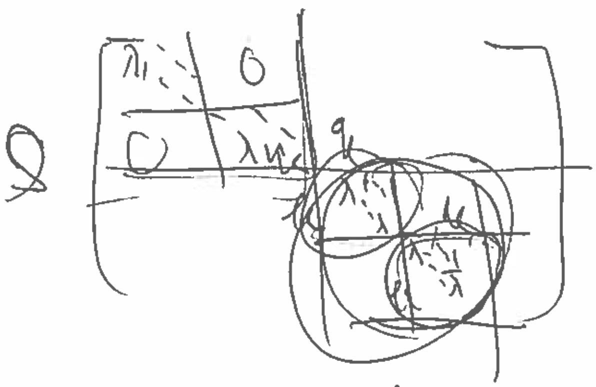
  Second if $S = S_1 + iS_2$ for some real matrix S_1, S_2, show that there is $\hat{S} = S_1 + rS_2$ for some real number r such that $\hat{S}$ is invertible so that $\hat{S}^{-1}A\hat{S}$ has the desired form.

$\hat{S}^{-1} S^{-1} A S \hat{S}$

f $\det(zI - A) = z^n + \cdots$

$z^2+1 = (z+i)(z-i)$

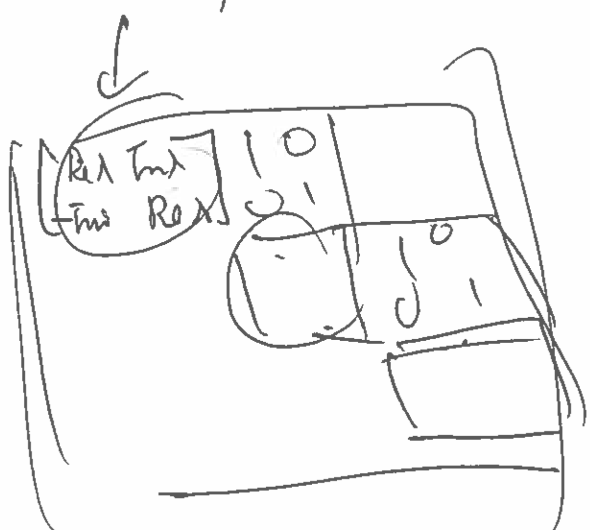
$\cancel{z^2+a}$ $f(z)$ has $\lambda \in \mathbb{C}\setminus\mathbb{R}$ as a zero.



$\overline{f(\lambda)} = \overline{0}$.

$\|$

$f(\bar\lambda) = 0$



2 Unitary similarity, unitary equivalence, and consequences

2.1 Unitary space and unitary matrices

Definition 2.1 *Let* $\mathbf{x} = (x_1, \ldots, x_n)^t, \mathbf{y} = (y_1, \ldots, y_n)^t$. *Define the inner product of* $\mathbf{x}$ *and* $\mathbf{y}$ *by*

$$\boxed{\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{j=1}^{n} x_j \bar{y}_j.}$$

In $\mathbb{C}^2$, $x = \begin{pmatrix} 1 \\ i \end{pmatrix} = y$

$\cancel{y = \begin{pmatrix} 1 \\ i \end{pmatrix}}$, $\langle x, y \rangle = 1 \cdot 1 + (i)(\bar{i}) = 2$.

The vectors are orthogonal if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$. *A set* $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ *is orthonormal if* $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \delta_{ij}$. *A matrix* $U \in M_n$ *with orthonormal columns is a unitary matrix, i.e.,* $U^*U = I_n$. *A basis for* $\mathbb{C}^n$ *is an orthonormal basis if it consists of orthonormal vectors.*

Some basic properties For any $a, b \in \mathbb{C}, \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{C}^n$,

- $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$.
- $\langle a\mathbf{x} + b\mathbf{y}, \mathbf{z} \rangle = a\langle \mathbf{x}, \mathbf{y} \rangle + b\langle \mathbf{x}, \mathbf{z} \rangle$,
- $\boxed{\langle \mathbf{x}, a\mathbf{y} + b\mathbf{z} \rangle = \bar{a}\langle \mathbf{x}, \mathbf{y} \rangle + \bar{b}\langle \mathbf{x}, \mathbf{z} \rangle}$,
- $\langle \mathbf{x}, \mathbf{x} \rangle = \sum_{j=1} |x_j|^2$ if $\mathbf{x} = (x_1, \ldots, x_n)^t$.
- $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, where the equality holds if and only if $\mathbf{x} = \mathbf{0}$.

$= x_1\bar{x}_1 + \cdots + x_n\bar{x}_n = \sum |x_i|^2$

$\langle x, ay + bz \rangle = \overline{\langle ay + bz, x \rangle} = \overline{a\langle y, x \rangle + b\langle z, x \rangle} = \bar{a}\,\overline{\langle y, x \rangle} + \bar{b}\,\overline{\langle z, x \rangle} = \bar{a}\langle x, y \rangle + \bar{b}\langle x, z \rangle$

$AS = S\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

$\Uparrow$ invertible

$S = [x_1 | \cdots | x_n]$ form a basis for $\mathbb{C}^n$.

$S = \begin{bmatrix} 1 & 1 \\ 0 & \varepsilon \end{bmatrix}$

In $\mathbb{C}^n$, cannot define

$$\left\langle \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right\rangle = \sum x_i y_i$$

$$\left\langle \begin{pmatrix} 1 \\ i \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} \right\rangle = 0$$

In $\mathbb{R}^n$, we want an orthonormal basis

$\mathcal{B} = \{u_1, \ldots, u_n\}$ s.t.

$u_i^t u_i = \|u_i\| = 1$

$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $\|x\| = (\sum x_i^2)^{1/2} = (x^t x)^{1/2}$

$u_i^t u_j = 0$ ($i \neq j$)

$= \sum_{p=1}^{n} x_p y_p$

$u_i = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $u_j = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$U = [u_1 | \cdots | u_n]$

$U^t U = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ & \ddots & & 0 \\ 0 & & & 1 \end{bmatrix}$

orthogonal matrix

In $\mathbb{R}^2$ $\begin{bmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$?

$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$

Let $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ $z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$.

To prove $\langle x,y \rangle = \overline{\langle y,x \rangle}$

$$\langle x,y \rangle = x_1\bar{y}_1 + \cdots + x_n\bar{y}_n$$

$$\langle y,x \rangle = y_1\bar{x}_1 + \cdots + y_n\bar{x}_n$$

$$\therefore \overline{\langle y,x \rangle} = \overline{y_1\bar{x}_1 + \cdots + y_n\bar{x}_n}$$

$$= \bar{y}_1\bar{\bar{x}}_1 + \bar{y}_2\bar{\bar{x}}_2 + \cdots + \bar{y}_n\bar{\bar{x}}_n$$

$$= \bar{y}_1 x_1 + \cdots + \bar{y}_n x_n$$

$$= \langle x,y \rangle.$$

$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

$\bar{\bar{z}}_1 = z_1$

To prove $\langle ax + by, z \rangle = a\langle x,y \rangle + b\langle x,z \rangle$

$$ax+by = \begin{pmatrix} ax_1 + by_1 \\ \vdots \\ ax_n + by_n \end{pmatrix}$$

$$\therefore \langle ax+by, z \rangle = (ax_1+by_1)\bar{z}_1 + \cdots + (a_nx_n+by_n)\bar{z}_n$$

$$a\langle x,z \rangle + b\langle y,z \rangle = (ax_1\bar{z}_1 + \cdots + ax_n\bar{z}_n) + (by_1\bar{z}_1 + \cdots + by_n\bar{z}_n)$$

$\langle x, x \rangle = \|x\|^2$

If $\langle x, y \rangle = 0$ then x & y in $\mathbb{C}^n$ are orthogonal.

Theorem 2.2 (a) *Orthonormal sets in $\mathbb{C}^n$ are linearly independent. If $\{u_1, \ldots, u_n\}$ is an orthonormal basis for $\mathbb{C}^n$, then for any $v \in \mathbb{C}^n$, $v = c_1 u_1 + \cdots + c_n u_n$ with $c_j = \langle v, u_j \rangle$ for $j = 1, \ldots, n$.*

(b) [Gram Schmidt process] *Suppose $A = M_{n,m}$ has linearly independent columns. Then $A = VR$ such that $V \in M_{n,m}$ has orthonormal column, and $R \in M_m$ is upper triangular.*

(c) *Every linearly independent (orthonormal) subset of $\mathbb{C}^n$ can be extended to a (an orthonormal) basis.*

(a) If $\{u_1, \ldots, u_k\}$ is orthonormal, then it is linearly independent

i.e. ~~$u_i \cdot u_j$~~

$$\langle u_i, u_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$