$\{v_1 \cdots v_k\} \subseteq \mathbb{C}^n, \quad \langle v_i, v_j \rangle = \begin{cases} 1 & i \neq j \\ 0 & i \neq j \end{cases}$

Theorem 2.2 (a) *Orthonormal sets in $\mathbb{C}^n$ are linearly independent. If $\{u_1, \ldots, u_n\}$ is an orthonormal basis for $\mathbb{C}^n$, then for any $v \in \mathbb{C}^n$, $v = c_1u_1 + \cdots + c_nu_n$ with $c_j = \langle v, u_j \rangle$ for $j = 1, \ldots, n$.*

$n \times m \quad n \times m \quad m \times m$

(b) [Gram Schmidt process] *Suppose $A = M_{n,m}$ has linearly independent columns. Then $\boxed{A = VR}$ such that $V \in M_{n,m}$ has orthonormal column, and $R \in M_m$ is upper triangular.* $S^{-1}AS = [\ ]$

(c) *Every linearly independent (orthonormal) subset of $\mathbb{C}^n$ can be extended to a (an orthonormal) basis.*

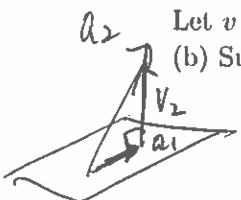
Proof. (a) Suppose $\{u_1, \ldots, u_k\}$ is orthonormal, and $\overset{\in \mathbb{C}}{a_1}u_1 + \cdots + \overset{\in \mathbb{C}}{a_k}u_k = 0$. Then

$$\mathbb{C} \ni 0 = \langle \sum a_j u_j, u_i \rangle = c_i \qquad \text{for all } i = 1, \ldots, k.$$

$0 = \langle \alpha_1 u_1 + \cdots + \alpha_k u_k, u_i \rangle = \alpha_1 \cancel{\langle u_1, u_i \rangle} + \cdots + \boxed{\alpha_i \langle u_i, u_i \rangle}^{1} + \cdots + \alpha_k \langle u_k, u_i \rangle$

Let $v = c_1u_1 \ldots, c_nu_n$. Then $c_i = \langle v, u_i \rangle$ for all $i = 1, \ldots, n$.

(b) Suppose A has column $\boxed{a_1, \ldots, a_m}$. Let



$$\boxed{u_1 = a_1} \quad u_2 = a_2 - \frac{\langle a_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1, \quad u_3 = a_3 - \frac{\langle a_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle a_3, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2, \ldots.$$

Then set $v_j = u_j / \|u_j\|$ for $j = 1, \ldots, m$, and V be $n \times m$ with columns $v_1, \ldots, v_m$. We have $A = VR$, where $R = V^*A$ is upper triangular.

(c) Let $\{v_1, \ldots, v_k\}$ be linearly independent. Take the pivoting columns of $[v_1 \cdots v_k \; e_1 \cdots e_n]$ to form an invertible matrix, and then apply Gram Schmidt process.

$A = [a_1 | \cdots | a_m]$.

$u_1 = a_1 \qquad u_2 = a_2 - c_1 u_1$

$\langle u_2, u_1 \rangle = 0$, i.e., $\overset{0}{\langle u_2, u_1 \rangle} = \langle a_2, u_1 \rangle - c_1 \langle u_1, u_1 \rangle$

$\therefore c_1 = \dfrac{\langle a_2, u_1 \rangle}{\langle u_1, u_1 \rangle}$

$a_1 = r_{11} v_1$

$a_2 = r_{12} v_1 + r_{22} v_2$

$a_3 = r_{13} v_1 + r_{23} v_2 + r_{33} v_3$

$u_3 = a_3 - c_1 u_1 - c_2 u_2$

$0 = \langle u_3, u_1 \rangle = \langle a_3, u_1 \rangle - c_1 \langle u_1, u_1 \rangle - c_2 \cancel{\langle u_2, u_1 \rangle}$

$\therefore c_1 = \dfrac{\langle a_3, u_1 \rangle}{\langle u_1, u_1 \rangle}$

$$[a_1 | a_2 | a_3] = [v_1 v_2 v_3] \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix}$$

$0 = \langle u_3, u_2 \rangle = \langle a_3, u_2 \rangle - c_1 \overset{0}{\langle u_1, u_2 \rangle} - c_2 \langle u_2, u_2 \rangle$

Then

$A = VR$

$\overset{m \times n}{V^*} \overset{n \times m}{A} = V^*VR = R.$

Example

Give $v_1 = \dfrac{\begin{bmatrix} 1 \\ -i \\ i \end{bmatrix}}{\sqrt{3}}$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ -i & 0 & 1 & 0 \\ i & 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -i & 1 & 0 \\ i & 0 & 1 \end{bmatrix}$$

u_1^*

$$u_1 = \begin{bmatrix} 1 \\ -i \\ i \end{bmatrix}, \quad \underline{\underline{u_2}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{[1\ i\ -i]\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}{[1\ i\ -i]\begin{bmatrix} 1 \\ -i \\ i \end{bmatrix}} \begin{bmatrix} 1 \\ -i \\ i \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{i}{3} \begin{bmatrix} 1 \\ -i \\ i \end{bmatrix}$$

$$\hat{u}_2 = \underline{\underline{3u_2}} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} - \begin{bmatrix} i \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -i \\ 2 \\ 1 \end{bmatrix}$$

$$\langle u_1, \hat{u}_2 \rangle = \underline{[i\ 2\ 1]} \underline{\begin{bmatrix} 1 \\ -i \\ i \end{bmatrix}} = 0$$

$$u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{[1\ i\ -i]\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}{[1\ i\ -i]\begin{bmatrix} 1 \\ -i \\ i \end{bmatrix}} \begin{bmatrix} 1 \\ -i \\ i \end{bmatrix} - \frac{[i\ 2\ 1]\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}{[i\ 2\ 1]\begin{bmatrix} -i \\ 2 \\ 1 \end{bmatrix}} \begin{bmatrix} -i \\ 2 \\ 1 \end{bmatrix}$$

8

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \frac{i}{3} \begin{bmatrix} 1 \\ -i \\ i \end{bmatrix} - \frac{1}{6} \begin{bmatrix} -i \\ 2 \\ 1 \end{bmatrix}$$

$$6u_3 = \begin{bmatrix} 3i \\ 0 \\ 3 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 1 \\ -i \\ i \end{bmatrix}, \begin{bmatrix} -i \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\mathcal{B} = \left\{ \frac{\begin{bmatrix} 1 \\ -i \\ i \end{bmatrix}}{\sqrt{3}}, \frac{\begin{bmatrix} -i \\ 2 \\ 1 \end{bmatrix}}{\sqrt{6}}, \frac{\begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix}}{\sqrt{2}} \right\}$$ orthonormal set basis

---

$$V = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-i}{\sqrt{6}} & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{i}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Check $U^*U = I_3$

$A = VR$ with $R = V^*A$.

QR factorization & solving equations

$$\overset{n\times m}{A}\,\overset{m\times 1}{x} = \overset{n\times 1}{0}$$

$$\overset{n\times m}{V}\,\overset{m\times m}{R}\,\overset{m\times 1}{x} = \overset{n\times 1}{0}$$

$$\overset{m\times n}{V^*}\,(\overset{n\times m}{V}\,\overset{m\times m}{R}\,\overset{m\times 1}{x}) = \overset{m\times n}{V^*}\,\overset{n\times 1}{0}$$

$$Rx = 0$$

$$\begin{pmatrix} \diagdown & \\ 0 & \diagdown \end{pmatrix} x = 0$$

$\langle x, y \rangle =$

$x_1\bar{y}_1 + x_2\bar{y}_2 + \cdots$

$$\begin{bmatrix} \bar{v}_{11} & \bar{v}_{21} & \cdots & \bar{v}_{n1} \\ \bar{v}_{12} & \bar{v}_{22} & \cdots & \bar{v}_{n2} \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & \vdots \\ \vdots & \vdots \\ v_{n1} & v_{n2} \end{bmatrix}$$

$\langle v_1 | v_1 \rangle = 1$

$$A = \left[a_1 \middle| a_2 \right] \overset{2}{=} n \left[v_1 \middle| v_2 \right] \begin{bmatrix} \|a_1\| & x \\ 0 & x \end{bmatrix}$$

$$v_1 = \frac{a_1}{\|a_1\|}$$

$$v_2 =$$

$$\hat{A} = \left[a_2 \middle| a_1 \right] = \left[\hat{v}_1 \middle| \hat{v}_2 \right] \begin{bmatrix} \|a_2\| & x \\ 0 & x \end{bmatrix}$$

$$A = \left[a_1 | a_2 | e_3 | e_4 | e_5 \right] = V R$$

$$\hat{A} = \left[\hat{a}_1 | \hat{a}_2 | e_3 | e_4 | e_5 \right] = \hat{V} \hat{R}$$

Suppose $\mathcal{B} = \{v_1, \dots, v_n\}$ is an o.n. basis for $\mathbb{C}^n$.

Then for every $x \in \mathbb{C}^n$

$$x = c_1 v_1 + \cdots + c_n v_n = \underline{\left[v_1 | \cdots | v_n \right]} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$V^* x = V^* V \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

$$c_1 = \langle x, v_1 \rangle$$

$$\vdots$$

$$c_n = \langle x, v_n \rangle$$

2.2 Schur Triangularization

Theorem 2.3 *For every matrix $A \in M_n$ with eigenvalues $\mu_1, \ldots, \mu_n$ (listed with multipicities), there is a unitary U such that U^*AU is in upper (or lower) triangular form with diagonal entries $\mu_1, \ldots, \mu_n$.*

Proof. Similar to that of Theorem 1.8.

Definition 2.4 *Let $A \in M_n$. The matrix A is normal if $AA^* = A^*A$, the matrix A is Hermitian if $A = A^*$, the matrix A is positive semidefinite if $\mathbf{x}^*A\mathbf{x} \geq 0$ for all $x \in \mathbb{C}^n$, the matrix A the matrix A is positive definite if $\mathbf{x}^*A\mathbf{x} > 0$ for all nonzero $x \in \mathbb{C}^n$.*

Theorem 2.5 *Let $A \in M_n$.*

(a) *The matrix A is normal if and only if it is unitarily diagonalizable.*

(b) *The matrix A is unitary if and only if it is unitarily similar to a diagonal matrix with all eigenvalues having modulus 1.*

By induction: If I want λ_{i_1} to be the $(1,1)$ entry in the final triangular matrix, then we solve $(A - \lambda_{i_1} I)x = 0$ for some nonzero x.

We may assume $\|x\| = 1$, else replace x by $\frac{x}{\|x\|}$.

Extend $\frac{x}{\|x\|}$ to an orthonormal basis, so we get a unitary matrix U_1 with $\frac{x}{\|x\|}$ as the first column and

$$AU_1 = U_1 \left[\begin{array}{c|ccc} \lambda_{i_1} & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & A_1 & \\ 0 & & & \end{array}\right]$$

$$U_1^*AU_1 = \left[\begin{array}{c|c} \lambda_{i_1} & * \\ \hline 0 & A_1 \end{array}\right] \quad \text{satisfies}$$

$$\det(zI - U_1^*AU_1) = (z - \lambda_{i_1}) \underline{\det(zI_{n-1} - A_1)}$$

equals $(z-\lambda_1) \cdots \underset{(z-\lambda_{i_1})}{\cancel{}} \cdots (z-\lambda_n)$

By induction assumption, $\exists\ U_2 \in M_{n-1}$ s.t. $U_2^* A_1 U_2 = \begin{bmatrix} \lambda_{i_2} & & * \\ & \ddots & \\ 0 & & \lambda_{i_n} \end{bmatrix}$

$$\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & U_2^* \end{array}\right] U_1^* A U_1 \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & U_2 \end{array}\right] = \left[\begin{array}{c|ccc} \lambda_{i_1} & & * & \\ \hline 0 & \lambda_{i_2} & & * \\ & & \ddots & \\ & 0 & & \lambda_{i_n} \end{array}\right]$$

Thm Let $A \in M_n$ TFAE.

(a) $A^*A = AA^*$

(b) ~~A~~ $\exists$ unitary U s.t.

$U^*AU = D$. (Any U putting A in triangular form will ~~make~~ yield a diagonal form.)

Proof (b) $\Rightarrow$ (a).

Suppose $U^*AU = D$

$A = UDU^*$ $\quad A^* = (UDU^*)^* = (U^*)^* D^* U^*$

$= U\bar{D}U^*$

$\therefore$ $AA^* = (UDU^*)(U\bar{D}U^*)$

$= U\begin{pmatrix} |d_1|^2 & & 0 \\ & \ddots & \\ 0 & & |d_n|^2 \end{pmatrix} U^*$

$A^*A = (U\bar{D}U^*)(UDU^*) = U\begin{pmatrix} |d_1|^2 & & 0 \\ & \ddots & \\ 0 & & |d_n|^2 \end{pmatrix} U^*$.

(a) $\Rightarrow$ (b). Assume $AA^* = A^*A$.

Let $U^*AU = T = (t_{ij})$ upper triangular.

Note that $TT^* = U^*A\cancel{U U^*}A^*U = U^*AA^*U = U^*A^*AU$

$= U^*A^*U\,U^*AU = T^*T$.

$$\begin{pmatrix} t_{11} & t_{12} & t_{13} \\ 0 & t_{22} & t_{23} \\ 0 & 0 & t_{33} \end{pmatrix}\begin{pmatrix} \bar{t}_{11} & 0 & 0 \\ \bar{t}_{12} & \bar{t}_{22} & 0 \\ \bar{t}_{13} & \bar{t}_{23} & \bar{t}_{33} \end{pmatrix}$$

$$= \begin{pmatrix} \bar{t}_{11} & 0 & 0 \\ \bar{t}_{12} & \bar{t}_{22} & 0 \\ \bar{t}_{13} & \bar{t}_{23} & \bar{t}_{33} \end{pmatrix}\begin{pmatrix} t_{11} & t_{12} & t_{13} \\ 0 & t_{22} & t_{23} \\ 0 & 0 & t_{33} \end{pmatrix}$$

(1,1) entry $|t_{11}|^2 + |t_{12}|^2 + |t_{13}|^2 = |t_{11}|^2$

Comparing (k,k) entries for TT^* & T^*T.

~~for~~ for $k = 1, \ldots, n$, we see that

$T = \begin{pmatrix} t_{11} & & 0 \\ & \ddots & \\ 0 & & t_{nn} \end{pmatrix}$. $\square$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$A = A^* \text{ i.e., } \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{pmatrix} \bar{a}_{11} & \bar{a}_{21} \\ \bar{a}_{12} & \bar{a}_{22} \end{pmatrix}$$

$$\begin{bmatrix} 1 & 1+2i \\ 1-2i & -2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1-i & 1+4i \\ 1+i & 3 & 2+i \\ 1-4i & 2-i & 5 \end{bmatrix}$$

If $A = A^*$ then $AA^* = A^2 = A^*A$

If A is unitary then $A^*A = I = AA^*$

A is positive semidefinite if

$$[\bar{x}_1 \cdots \bar{x}_n]\, A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \geq 0 \quad \text{for any } \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{C}^n$$

Example

$$(\bar{x}_1 \ \bar{x}_2) \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= 2(x_1)\ 2\bar{x}_1 x_1 + \bar{x}_2 x_2 = 2|x_1|^2 + |x_2|^2 \geq 0$$

Example

$$[\bar{x}_1 \ \bar{x}_2] \begin{pmatrix} 2 & -i \\ i & 2 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= (\bar{x}_1 \bar{x}_1) \begin{bmatrix} 2x_1 - ix_2 \\ ix_1 + 2x_2 \end{bmatrix}$$

$$= \bar{x}_1 (2x_1 - ix_2) + \bar{x}_2 (ix_1 + 2x_2)$$

$$= 2|x_1|^2 - i\bar{x}_1 x_2 + i\bar{x}_2 x_1 + 2|x_1|^2$$

$$\geq 2|x_1|^2 + 2|x_2|^2 \geq 2|x_1 x_2| \geq 0$$

Hermitian not positive definite.

$$A = \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x^* A x = \underline{-1}$$

Not hermitian
not positive semi
definite

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} i & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} i & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= i$$

Thm $\underline{A \in M_n}$ is unitary $\Leftrightarrow U^*AU = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$

with $|d_i| = 1 \ \forall i = 1, \ldots, n$.

Proof: Suppose. $A^*A = I = AA^*$.

Then $U^*AU = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix} =: D$

& $DD^* = U^*AU\, U^*A^*U = U^*\cancel{A}A^*U$

$= I$

$$\begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix} \begin{pmatrix} \bar{d}_1 & & 0 \\ & \ddots & \\ 0 & & \bar{d}_n \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$\Rightarrow \quad |d_i| = 1 \quad \forall i = 1, \ldots, n$.

Assume $U^*AU = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$ with $|d_i| = 1$.

Then $U^*AU\, U^*A^*U = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix} \begin{pmatrix} \bar{d}_1 & & 0 \\ & \ddots & \\ 0 & & \bar{d}_n \end{pmatrix}$

$= I_n$

$\therefore \ U^*(AA^*)U = I_n$

$AA^* = U I_n U^* = I_n$.