Recall:

$\nearrow U^*U = I_n$

Let. $A \in M_n$. There is a unitary U s.t.

$$U^*AU = \begin{bmatrix} \ddots & * \\ 0 & \ddots \end{bmatrix}$$

A is normal $A^*A = A^*A \iff U^*AU = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$.

A is unitary $A^*A = I \iff U^*AU = \begin{bmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{bmatrix}$

with $|\mu_i| = 1 \quad \forall i$

A is Hermitian, $A = A^*$ $\qquad A = \begin{bmatrix} 1 & 1+i & 1-i \\ 1-i & 2 & 3+i \\ 1+i & 3-i & 3 \end{bmatrix}$

$\iff$???

A is positive ~~semidefinite~~ semidefinite, i.e., $x^*Ax \geq 0 \ \forall x \in \mathbb{C}^n$

$\iff$???

$x \in \mathbb{C}^n$, U unitary

$\langle x, x\rangle = x^* x$

$\langle Ux, Ux\rangle = x^* U^* U x = x^* I x = x^* x$.

Theorem 2.6 *Let $A \in M_n$. The following are equivalent.*

(a) *A is Hermitian.* $A = A^*$

(b) *A is unitarily similar to a real diagonal matrix.*

(c) $\mathbf{x}^* A\mathbf{x} \in \mathbb{R}$ *for all* $\mathbf{x} \in \mathbb{C}^n$.

(a) $\Rightarrow$ (b). $A = A^* \Rightarrow AA^* = A^2 = A^*A$ $\therefore$ $U^*AU = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$.

Now $\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} = U^*AU = U^*A^*U = (U^*AU)^* = \begin{pmatrix} \bar\lambda_1 & & 0 \\ & \ddots & \\ 0 & & \bar\lambda_n \end{pmatrix}$.

$\therefore$ $\lambda_i = \bar\lambda_i$, i.e., $\lambda_i \in \mathbb{R}$ $\forall i = 1, \dots, n$.

(b) $\Rightarrow$ (c) Assume $A = U \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} U^*$, $\lambda_i \in \mathbb{R}$.

Then for any $x \in \mathbb{C}^n$, let $U^* x = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{C}^n$.

we have $x^*Ax = x^*U \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} U^* x$

$= (\bar y_1 \cdots \bar y_n) \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$y^* = x^*(U^*)^* = x^* U$

$= (\bar y_1 \cdots \bar y_n) \begin{pmatrix} \lambda_1 y_1 \\ \vdots \\ \lambda_n y_n \end{pmatrix} = \lambda_1 |y_1|^2 + \cdots + \lambda_n |y_n|^2 \in \mathbb{R}$.

(c) $\Rightarrow$ (a) Assume $x^*Ax \in \mathbb{R}$ for all $x \in \mathbb{C}^n$.

Let $A = H + i\underbrace{G}_{}$, $H = \dfrac{A + A^*}{2}$, $G = \dfrac{A - A^*}{2i}$ {Hermitian decomposition}

Skew-Hermitian

Hermitian Hermitian

We only need to show $G = 0$.

If G is not zero, then

$U^*GU = \begin{bmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{bmatrix}$; $\mu_i \in \mathbb{R}$. say, $\mu_1 \neq 0$.

Let x be the first column of U.

Then $x^*Ax = x^*Hx + ix^*Gx$

$= h + i\mu_1 \notin \mathbb{R}$, !!!

$H^* = \left(\dfrac{A + A^*}{2}\right)^* = \dfrac{A^* + (A^*)^*}{2} = \dfrac{A^* + A}{2} = H$

$G^* = \left[\left(\dfrac{I}{2i}\right)(A - A^*)\right]^*$

$= (A - A^*)^* \left(\dfrac{I}{2i}\right)^* = (A^* - A)\left(\dfrac{-I}{2i}\right)$

$= \dfrac{A - A^*}{2i} = G$.

Theorem 2.7 *Let $A \in M_n$. The following are equivalent.*

(a) *A is positive semidefinite.* $x^*Ax \geq 0 \quad \forall x \in \mathbb{C}^n$.

(b) *A is unitarily similar to a real diagonal matrix with nonnegative diagonal entries.*

(c) *$A = B^*B$ for some $B \in M_n$. (We can choose B so that $B = B^*$.)*

$= B^2$ if $B = B^*$. $A = \sqrt{A}^2$ if $B = \sqrt{A}$ is psd.

Proof: (a) $\Rightarrow$ (b). $x^*Ax \geq 0 \Rightarrow$ $x^*Ax \in \mathbb{R} \Rightarrow A$ is Hermitian

$\Rightarrow A = U \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} U^*$ for some $\lambda_1, \dots, \lambda_n \in \mathbb{R}$.

Suppose $\lambda_i < 0$. Now

Let x be the ith column of U. $\underline{U^*AU} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

Then $x^*Ax = \lambda_i < 0$!!!

(b) $\Rightarrow$ (c). Suppose $A = U^* \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} U$, $\lambda_i \geq 0$.

Then for $D = \begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{bmatrix}$.

We have $A = U^* \begin{bmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{bmatrix} \begin{bmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{bmatrix} U$.

$= B^*B = C^2$

with $B = \begin{bmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{bmatrix} U$, $C = U^* \begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{bmatrix} U$

(c) $\Rightarrow$ (a). Suppose $A = B^*B$.
Let $x \in \mathbb{C}^n$. Then
$x^*Ax = x^*B^*Bx$
$= y^*y \geq 0$ if $\underline{\underline{y = Bx}}$

For any $A \in M_n$ we can write $A = H + iG$ with $H = (A + A^*)/2$ and $G = (A - A^*)/(2i)$. This is known as the Hermitian or Cartesian decomposition.

$y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ $\langle y, y \rangle = \sum_{i=1}^{n} y_i \bar{y}_i = \sum |y_i|^2 \geq 0$

2.3 Spetch's theorem and Commuting families

There is no easy canonical form under unitary similarity. [1] How to determine two matrices are unitarily similar?

Definition 2.8 *Let $\{X, Y\} \subseteq M_n$. A word $W(X, Y)$ in X and Y of length m is a product of m matrices chosen from $\{X, Y\}$ (with repetition).*

XXYYXY

X, Y, XX, XY, YX, YY

Theorem 2.9 *Let $A, B \in M_n$.*

(a) *If A and B are unitarily similar, then $\operatorname{tr}(W(A, A^*)) = \operatorname{tr}(W(B, B^*))$ for all words $W(X, Y)$.*

(b) If *$\operatorname{tr}(W(A, A^*)) = \operatorname{tr}(W(B, B^*))$ for all words $W(X, Y)$ of length $2n^2$, then A and B are unitarily similar.*

(a) For example

$\operatorname{tr} A = \operatorname{tr} B$

$\operatorname{tr} A^* = \operatorname{tr} B^*$

$\operatorname{tr} A^2 = \operatorname{tr} B^2$

$\operatorname{tr} AA^* = \operatorname{tr} BB^*$

$\operatorname{tr} A^*A = \operatorname{tr} B^*B$

$\operatorname{tr} A^*A^* = \operatorname{tr} B^*B^*$

n	length of words	# of words
2	8	$2^8+2^7+\dots+2^0$
3	18	$2^{18}+\dots+2^0$

For $n=2$,

$U^*AU = \begin{bmatrix} \lambda_1 & a \\ 0 & \lambda_2 \end{bmatrix}$, $V^*BV = \begin{bmatrix} \lambda_1 & b \\ 0 & \lambda_2 \end{bmatrix}$

$A \approx U^*BU \Leftrightarrow$

$\Leftrightarrow \operatorname{tr} A = \operatorname{tr} B$, $\operatorname{tr} A^2 = \operatorname{tr} B^2$

$\operatorname{tr} AA^* = \operatorname{tr} BB^*$

$\cancel{\operatorname{tr} AA^*AA^* = \operatorname{tr} BB^*BB^*}$

Proof: Assume $B = U^*AU$.

$W(X,Y) = X^{n_1}Y^{n_2} \cdots X^{n_{2k+1}}Y^{n_{2k+2}}$

$n_i \geq 0$

Then

$\operatorname{tr} W(B, B^*)$

$= \operatorname{tr}\left(B^{n_1}(B^*)^{n_2} \cdots B^{n_{2k+1}} B^{*n_{2k+2}}\right)$

$= \operatorname{tr}\left((U^*AU)^{n_1}(U^*A^*U)^{n_2} \cdots (U^*A^*U)^{n_{2k+2}}\right)$

$= \operatorname{tr} U^*(A^{n_1} \cdots A^{*n_{2k+2}})U$

$= \operatorname{tr}(A^{n_1} \cdots A^{*n_{2k+2}})UU^*$

$= \operatorname{tr} W(A, A^*)$.

$\operatorname{tr}(XY) = \operatorname{tr}(YX)$

[1] Helene Shapiro, A survey of canonical forms and invariants for unitary similarity, Linear Algebra Appl. 147 (1991), 101-167.

Definition 2.10 *A family $\mathcal{F} \subseteq M_n$ is a commuting family if every pair of matrices $X, Y \in \mathcal{F}$ commute, i.e., $XY = YX$.*

Lemma 2.11 *Let $\mathcal{F} \subseteq M_n$ be a commuting family. Then there a unit vector $v \in \mathbb{C}^n$ such that v is an eigenvector for every $A \in \mathcal{F}$.*

$Av = \lambda_A v$

Theorem 2.12 *Let $\mathcal{F} \subseteq M_n$. Then there is an invertible matrix $S \in M_n$ such that $S^{-1}AS$ is in upper triangular form. The matrix S can be chosen to be unitary.*

Proof. Similar to the proof of Theorem 1.8. ☐

$A_i = S^{-1} \left[\begin{array}{c|c} * & * \\ \hline 0 & B \end{array}\right] S$

Corollary 2.13 *Suppose $\mathcal{F} \subseteq M_n$ is a commuting family of normal matrices. Then there is a unitary matrix $U \in M_n$ such that U^*AU is in diagonal form.*

Let S have first colum v so that $Av = \lambda_A v$ $\forall A \in \mathcal{F}$.

Then

$$\underbrace{A S}_{[v|v_2|\cdots|v_n]} = S \left[\begin{array}{c|c} \lambda_A & * \\ \hline 0 & \\ \vdots & B_A \\ 0 & \end{array}\right]$$

Then $XY = YX$ $\forall X, Y \in \mathcal{F}$ implies.

$$X = S \left[\begin{array}{c|c} \lambda_X & * \\ \hline 0 & B_X \end{array}\right] S^{-1} \qquad Y = S \left[\begin{array}{c|c} \lambda_Y & * \\ \hline 0 & B_Y \end{array}\right] S^{-1}$$

Satisfying

$$XY = YX = S \left[\begin{array}{c|c} \lambda_Y\lambda_X & * \\ \hline 0 & B_Y B_X \end{array}\right] S^{-1}$$

$$XY = S \left[\begin{array}{c|c} \lambda_X\lambda_Y & * \\ \hline 0 & B_X B_Y \end{array}\right] S^{-1}$$

i.e., $B_X B_Y = B_Y B_X$ $\forall X, Y \in \mathcal{F}$.

By induction (on n) assumption there is S_2 s.t.

$$S_2^{-1} B_X S_2 = \begin{bmatrix} \diagdown & \\ 0 & \diagdown \end{bmatrix} \quad \forall X \in \mathcal{F}.$$

Then

$$\underbrace{\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & S_2^{-1} \end{array}\right] S^{-1}} X \underbrace{S \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & S_2 \end{array}\right]} = \begin{bmatrix} \diagdown & * \\ 0 & \diagdown \end{bmatrix}$$

$$\forall X \in \mathcal{F}.$$