Lemma 2.11 *Let $\mathcal{F} \subseteq M_n$ be a commuting family. Then there a unit vector $v \in \mathbb{C}^n$ such that v is an eigenvector for every $A \in \mathcal{F}$.*

$Av = \lambda_A v$

Proof: Consider $V \subseteq \mathbb{C}^n$ s.t. $A(V) \subseteq V \quad \forall A \in \mathcal{F}$.
Want to show that $\exists$ such a $V \subseteq \mathbb{C}^n$ of dim 1.

Let V be such a subspace of $\mathbb{C}^n$ with minimum positive dimension.
Will show ~~that~~ this the dimension must be 1.
Assume not. Then $A(V) \subseteq V$ has min dimension $k > 1$.
for all $A \in \mathcal{F}$

$\therefore \exists S$ invertible s.t. $S^{-1}AS = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$

Let $A_0 = S\begin{bmatrix} \hat{A}_0 & * \\ 0 & * \end{bmatrix}S^{-1} \in \mathcal{F}$. and let $\hat{A}_0 u = \lambda u$

Note: If $[v_1 \cdots v_k]$ is a basis for V

$A[v_1 \cdots v_k | v_{k+1} \cdots v_n]$

$= [v_1 \cdots v_k | v_{k+1} \cdots v_n]\begin{bmatrix} r_{11} & r_{12} & \cdots \\ \vdots & & \\ 0 & & \end{bmatrix}$

$S^{-1}AS = \begin{array}{c} \\ k\{ \end{array}\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$, $\underbrace{}_{k}$

so that

$S\begin{bmatrix} u \\ 0 \end{bmatrix} \in V.$

and $A_0 S\begin{bmatrix} u \\ 0 \end{bmatrix} = S\begin{bmatrix} \hat{A}_0 & * \\ 0 & * \end{bmatrix}S^{-1}S\begin{bmatrix} u \\ 0 \end{bmatrix}$

$= S\begin{bmatrix} \hat{A}_0 u \\ 0 \end{bmatrix} = S\begin{bmatrix} \lambda u \\ 0 \end{bmatrix} \in V$

Here I choose A_0 so that not all vectors in V are eigenvectors of A_0.

Consider ~~$\ker(A_0 -$~~ $W = \{ w \in V : A_0 w = \lambda w \}$

Then $A(W) \subseteq V \quad \forall A \in \mathcal{F}$.

W has positive dimension $\because S\binom{u}{0} \in W$.

and $W \subsetneqq V$.

Claim: for any $w \in W$, $A \in \mathcal{F}$:

$AA_0 = A_0 A \Rightarrow A_0(Aw) = AA_0 w = A\lambda w = \lambda(Aw)$

So $A(W) \subseteq W \quad \forall A \in \mathcal{F}$!!! $\therefore$ b/c W has smaller dimension !!!

Recall $\mathcal{B} = \{E_{11}, E_{12}, \ldots, E_{mn}\}$ is the standard basis for $M_{m,n}$.

$E_{ij} = i\begin{bmatrix} 0 & & 0 \\ & 1 & \\ 0 & & 0 \end{bmatrix}$

2.4 Singular decomposition and polar decomposition

Lemma 2.14 *Let A be a nonzero $m \times n$ matrix, and $u \in \mathbb{C}^m, v \in \mathbb{C}^n$ be unit vectors such that $|u^*Av|$ attains the maximum value. Suppose $U \in M_m$ and $V \in M_n$ are unitary matrices with u and v as the first columns, respectively. Then $U^*AV = \begin{pmatrix} u^*Av & 0 \\ 0 & A_1 \end{pmatrix}$.*

Proof

Note that if $\max|a_{ij}| = M$, then $|u^*Av| \le Mmn$.

Let $\mathcal{U}^*AV = \begin{bmatrix} u^*Av & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & * & \\ a_{m1} & & & \end{bmatrix}$

Assume $x = \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}$ has non-zero entry a_{i1}, $i > 1$.

Consider $u_0 = \dfrac{x}{\|x\|}$ and let $\hat{u} = \mathcal{U}u_0 = \mathcal{U}\dfrac{x}{\|x\|}$ is a unit vector.

$$|\hat{u}^*Av| = |u_0^*\mathcal{U}^*Av| = \left|\frac{x^*}{\|x\|}(\mathcal{U}^*AV)\right| = \left|\frac{[\bar{a}_{11} \cdots \bar{a}_{m1}]\begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}}{\|x\|}\right|$$

$$= \frac{\left(\sum_{i=1}^m |a_{i1}|^2\right)}{(\quad)^{1/2}} = \sqrt{\sum_{i=1}^m |a_{i1}|^2} > |a_{11}|, \ !!!$$

Similarly, if $a_{1j} \neq 0$ for $j > 1$

then we can let $v_0 = \dfrac{\begin{bmatrix} \bar{a}_{11} \\ \bar{a}_{12} \\ \vdots \\ \bar{a}_{1n} \end{bmatrix}}{\|[\quad]\|}$ & $\hat{v} = Vv_0$.

Then $u^*A\hat{v} = \sqrt{\sum_{j=1}^n |a_{1j}|^2} > |a_{11}|. \ !!!$ ▨

Theorem 2.15 *Let A be an $m \times n$ matrix. Then there are unitary matrices $U \in M_m, V \in M_n$ such that*

$$U^*AV = D = \sum_{j=1}^{k} s_j E_{jj}.$$

$$\begin{pmatrix} u_1^* \\ \vdots \\ u_m^* \end{pmatrix} A \begin{pmatrix} v_1 \cdots v_n \end{pmatrix} = \begin{array}{c} \\ k \\ \\ \end{array} \begin{array}{cc} k & n-k \\ \left[\begin{array}{c|c} s_1 \; s_2 \; \ddots \; s_k & 0 \\ \hline 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ \\ m-k \end{array}$$

$s_1 \geq \cdots \geq s_k > 0$ are the singular values of A

$\mathcal{V} = \{v_1, \ldots, v_n\}$ basis for $\mathbb{C}^n$

T $\quad \mathcal{U} = \{u_1, \ldots, u_m\}$ basis for $\mathbb{C}^m$

$Tx = Ax$

$$[T]_{\mathcal{U},\mathcal{V}} = \left(\begin{array}{c|c} s_1 \ddots s_k & 0 \\ \hline 0 & 0 \end{array}\right)$$

$T(v_1) = s_1 u_1, \quad \cdots \quad T(v_k) = s_k u_k.$

$T(v_j) = 0 \quad j = k+1, \ldots, n.$

<u>Proof of Theorem 2.15</u>. Let $A \in M_{m,n}$.

Assume $A \neq 0$.

Then there are $u \in \mathbb{C}^m$, $v \in \mathbb{C}^n$ unit vectors such that

$|u^*Av|$ is max.

We can replace v by $e^{it}v$ so that

$|u^*Av| = u^*Av = s_1$

Then for unitary $U \in M_m$, $V \in M_n$ with u, v as first columns,

we have
$$U^*AV = \left[\begin{array}{c|c} s_1 & 0 \\ \hline 0 & A_1 \end{array}\right].$$

If $A_1 = 0$, we are done. If not, by induction / repeat the process to A_1 so that there are unitary $U_1 \in M_{m-1}$, $V_1 \in M_{n-1}$ s.t.

$$U_1^* A_1 V_1 = \left[\begin{array}{c|c} s_2 \ddots s_k & 0 \\ \hline 0 & 0 \end{array}\right] \begin{array}{l} k-1 \\ m-k-1 \end{array}$$

Then
$$\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & U_1^* \end{array}\right] U^*AV \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & V_1 \end{array}\right] = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & U_1^* \end{array}\right] \left[\begin{array}{c|c} s_1 & 0 \\ \hline 0 & A_1 \end{array}\right] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & V_1 \end{array}\right] = \left[\begin{array}{c|c|c} s_1 & 0 & 0 \\ \hline 0 & s_2 \ddots s_k & 0 \end{array}\right]$$

Remark 2.16 *Let U_1 be formed by the first k columns $u_1, \ldots, u_k$ of U and V_1 be formed by the first k columns $v_1, \ldots, v_k$ of V. Then*

$$A = U_1 \operatorname{diag}(s_1, \ldots, s_k) V_1^* = \sum_{j=1}^{k} s_j u_j v_j^*.$$

Note that $s_1^2, \ldots, s_k^2$ are the nonzero eigenvalues of AA^* and A^*A.

Let $\{v_1, \ldots, v_k\} \subseteq \mathbb{C}^n$ be an orthonormal set of eigenvectors corresponding to the nonzero eigenvalues $s_1^2, \ldots, s_k^2$ of A^*A. Let $u_j = Av_j/s_j$. Then $\{u_1, \ldots, u_k\} \subseteq \mathbb{C}^m$ is an orthonormal family such that $A = \sum_{j=1}^{k} s_j u_j v_j^*$.

Similarly, let $\{u_1, \ldots, u_k\} \subseteq \mathbb{C}^m$ be an orthonormal set of eigenvectors corresponding to the nonzero eigenvalues $s_1^2, \ldots, s_k^2$ of AA^*. Let $v_j = A^*u_j/s_j$. Then $\{v_1, \ldots, v_k\} \subseteq \mathbb{C}^n$ is an orthonormal family such that $A = \sum_{j=1}^{k} s_j u_j v_j^*$.

Corollary 2.17 *Let $A \in M_n$. Then $\boxed{A = U^*P = QV}$ such that $U, V \in M_n$ are unitary, and P, Q are positive semidefinite matrices with eigenvalues equal to the singular values of A.*

$1000 \times 2000 = 2 \times 10^6$

$|z| = \quad z = |z| e^{it}$

$A = U \begin{bmatrix} s_1 & & & 0 \\ & \ddots & & \\ & & s_k & \\ 0 & & & 0 \end{bmatrix} V^*$

$A = [u_1 \cdots u_m] \begin{bmatrix} s_1 & & & 0 \\ & \ddots & & \\ & & s_k & \\ 0 & & & 0 \end{bmatrix} \begin{bmatrix} v_1^* \\ \vdots \\ v_n^* \end{bmatrix}$

$= [u_1 \cdots u_n] \begin{bmatrix} s_1 v_1^* \\ \vdots \\ s_k u_k^* \end{bmatrix}$

$AA^* = U \begin{bmatrix} s_1^2 & & & \\ & \ddots & & \\ & & s_k^2 & \\ & & & 0 \end{bmatrix} U^*$

$A^*A = V \begin{bmatrix} s_1 & & \\ & \ddots & \\ & & s \end{bmatrix} U^* U \begin{bmatrix} s_1 & & \\ & \ddots & \\ & & \end{bmatrix} V^*$

$= V \begin{bmatrix} s_1^2 & & & 0 \\ & \ddots & & \\ & & s_k^2 & \\ 0 & & & 0 \end{bmatrix} V^*$

$n-k$