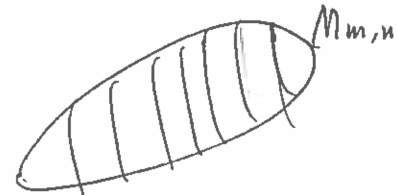

2.5 Other canonical forms

We have considered the canonical forms under similarity and unitary similarity. Here, we consider other canonical forms for different classes of matrices.

Unitary equivalence

- Two matrices $A, B \in M_{m,n}$ are unitarily equivalent if and only if there are unitary matrices $U \in M_m, V \in M_n$ such that $\boxed{U^*AV = B.}$ $\begin{bmatrix} s_1 \ddots s_k & 0 \\ 0 & 0 \end{bmatrix}$
- Every matrix $A \in M_{m,n}$ is unitarily equivalent to $\boxed{\sum_{j=1}^{k} s_j E_{jj}}$, where $s_1 \geq \cdots \geq s_k > 0$ are the nonzero singular values of A.
- Two matrices are unitarily equivalent if and only if they have the same singular values.

$A = U_1 \begin{bmatrix} s_1 \ddots s_k & 0 \\ 0 & 0 \end{bmatrix} V_1^*$

$B = U_2 \begin{bmatrix} s_1 \ddots s_k & 0 \\ 0 & 0 \end{bmatrix} V_2^*$

Equivalence

- Two matrices $A, B \in M_{m,n}$ are equivalent if there are invertible matrices $R \in M_m, S \in M_n$ such that $A = RBS$.
- Every matrix $A \in M_{m,n}$ is equivalent to $\sum_{j=1}^{k} E_{jj}$, where k is the rank of A.
- Two matrices are equivalent if they have the $\boxed{\text{same rank}}$.

Proof. Elementary row operations and elementary column operations. □

Reflexive $A \sim A$ $\because$ $A = I_m A I_n$

Symmetric $A \sim B$, i.e., $\exists$ unitary $U \in M_m, V \in M_n$, $U^* A V = B$, Then $A = \underline{U} B \underline{V^*}$ $\therefore B \sim A$.

Transitive If $A \sim B$, $B \sim C$ then $U_1^* A V_1 = B$

$U_2^* B V_2 = C$

then $C = U_2^* B V_2 = (U_2^* U_1^*) A (V_1 V_2)$

$\therefore A \sim C$

Suppose If $A = U_1 \begin{bmatrix} s_1 \ddots s_k & 0 \\ 0 & 0 \end{bmatrix} V_1^*$ $B = U_2 \begin{bmatrix} s_1 \ddots s_k & 0 \\ 0 & 0 \end{bmatrix} V_2^*$

If A & B have the same singular values, then

 then $U_1^* A V_1 = \begin{pmatrix} s_1 \ddots s_k & 0 \\ 0 & 0 \end{pmatrix} = U_2^* B V_2$

$\therefore$ $A = \underline{U_1 U_2^*} B \underline{V_2 V_1^*}$

Conversely,

If $A = U^* B V$, then

$$
\begin{aligned}
& AA^* \quad \cancel{\& \ U^* B V} \\
&= (U^* B V)(V^* B^* U) \\
&= U^*(BB^*)U \quad \text{~~tra~~ so that}
\end{aligned}
$$

AA^* & BB^* have the same ~~so~~ eigenvalues

$\therefore$ A & B have the same singular values.

Equivalence

$$
\underbrace{E_k \cdots E_1}_{R} A \overbrace{F_1 F_2 \cdots F_l}^{S} = \begin{bmatrix} 1 & 0 & 0 & & * \\ 0 & 1 & & & \\ 0 & 0 & 1 & & \end{bmatrix} = \left[\begin{array}{c|c} \begin{smallmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{smallmatrix} & 0 \\ \hline 0 & 0 \end{array}\right]
$$

For any $A, B \in M_{m,n}$

$$
R_1 A S_1 = \left[\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array}\right]
$$

$$
R_2 B S_2 = \left[\begin{array}{c|c} I_l & 0 \\ \hline 0 & 0 \end{array}\right]
$$

SVD Singular value decomposition

$$AA^* = V\begin{bmatrix} \begin{matrix} s_1^2 & 0 \\ 0 & s_k^2 \end{matrix} & 0 \\ 0 & 0 \end{bmatrix} V^*$$

A $m \times n$

$$A = U \begin{array}{c} \overbrace{}^{n-k} \\ \begin{bmatrix} \begin{matrix} s_1 & 0 \\ 0 & s_k \end{matrix} & 0 \\ 0 & 0 \end{bmatrix} \end{array} V^*_{I_{m-k}},$$

$U \in M_m$, $V \in M_n$ unitary

$s_1 \geq \cdots \geq s_k > 0$

nonzero singular values of A.

$$= s_1 u_1 v_1^* + \cdots + s_k u_k v_k^*$$

Polar Decomposition

$A \in M_n$

$$A = UDV^* = \underbrace{(UDU^*)}_{P}\underbrace{(UV^*)}_{W_1}$$

$$= \underbrace{UV^*}_{W_2}\underbrace{(VDV^*)}_{Q}$$

2, 3

Even for $A \in M_{m,n}$

$$A = U \begin{bmatrix} D & \overbrace{0}^{n-k} \\ 0 & 0 \end{bmatrix}_{2\times 2} V^* .$$

$m \times n$ 2×3

$$= U\begin{bmatrix} D & \overbrace{0}^{m-k} \\ 0 & 0 \end{bmatrix} U^* U \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} V^*$$

$$= U \overbrace{\begin{bmatrix} I_k & \\ & \end{bmatrix}}^{n} Q^* \left(V^* V \begin{bmatrix} I_k & \overbrace{0}^{n-k} \\ 0 & 0 \end{bmatrix} V^* \right)$$

2×3 3×3

Canonical form of Hermitian A under S^*AS:

$$A = A^* \Rightarrow A = U\begin{bmatrix} \lambda_1 & & & & & \\ & \lambda_p & & & & \\ & & \lambda_{p+1} & & & \\ & & & \lambda_{p+q} & & \\ & & & & 0 & \\ & & & & & 0 \end{bmatrix}U^*$$

$\lambda_1, \dots, \lambda_p > 0$.

$\lambda_{p+1}, \dots \lambda_{p+q} < 0$

$\lambda_{p+q+1} = \dots = \lambda_n = 0$

$$\overset{S^*}{\left[\begin{array}{c|c} \frac{1}{\sqrt{\lambda_1}} \ddots \frac{1}{\sqrt{|\lambda_{p+1}|}} & \\ \hline & I \end{array}\right]} \left[\begin{array}{c|c} \lambda_1 \ddots \lambda_p & 0 \\ \hline 0 & Q \end{array}\right] \overset{S}{\left[\begin{array}{c|c} \frac{1}{\sqrt{\lambda_1}} \ddots \frac{1}{\sqrt{\lambda_p}} \, \frac{1}{\sqrt{|\lambda_{p+1}|}} & \\ \hline & I \end{array}\right]}$$

$$= \left[\begin{array}{c|c|c} I_p & & \\ \hline & -I_q & \\ \hline & & O_{n-p-q} \end{array}\right]$$

$$\overset{S^*}{\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \overset{S}{\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\overset{S^*}{\left[\frac{1}{\sqrt{3}}\right]} [3] \overset{S}{\left[\frac{1}{\sqrt{3}}\right]} = [1]$$

$$\therefore S^*U^*AUS = S^*\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}S = \left[\begin{array}{c|c|c} I_p & 0 & 0 \\ \hline 0 & -I_q & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

Inertia of $A = A^* \in M_n$ is $(p, q, n-p-q)$

if A has p positive, q negative eigenvalues.

If A and B have the same inertia then

$$S^*AS = \begin{bmatrix} I_p & 0 & 0 \\ 0 & -I_q & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ for some invertible S & T.

& $$T^*BT = \begin{bmatrix} I_p & 0 & 0 \\ 0 & -I_q & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So

$$\cancel{A = (T^*)(S^*)^{-1}}$$

$$A = (S^*)^{-1} T$$

$$S^*AS = T^*BT$$

$$\therefore \; A = (S^*)^{-1} T^* B \boxed{T S^{-1}}$$

$$= R^* B R$$

$$R^* = (TS^{-1})^* = (S^*)^{-1} T^*$$

$A = A^*$, $B = B^*$

If $A = S^* B S$, then A, B have the same inertia ???

U^*AU

$$\begin{bmatrix} a_1 \cdots a_p & & \\ & a_{p+1} \cdots a_q & \\ & & \end{bmatrix}$$

$T^* S V^* B V S T$

$$S^* \begin{bmatrix} I_p & 0 & 0 \\ 0 & -I_q & 0 \\ 0 & 0 & 0 \end{bmatrix} S = \begin{bmatrix} I_{\hat p} & 0 & 0 \\ 0 & -I_{\hat q} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{———} (*)$$

Question (Extra Credit)

Suppose in $(*)$, $p + q = \hat{p} + \hat{q}$. Show that invertible S exists so that $(*)$ holds iff $p = \hat{p}$, $q = \hat{q}$.

Using a group of transformations to change $A \in M_{m,n}$

Example: Note that

$$G = \{ L : \; L_S \text{ is a linear transformation } : M_n \to M_n, \; L(A) = S^{-1}AS \}$$

~~Lform~~ is a subgroup.

(G0) If $L_1, L_2 \in G$, $\quad L_1 \circ L_2 \in G$

(G1) $\quad (L_1 L_2) L_3 = L_1 (L_2 L_3)$

(G2) There is $L \in G$ so that

$$L \circ \hat{L} = \hat{L} \circ L \text{ for all } \hat{L} \in G$$

$$L(A) = A = IAI$$

(G3) For every $\hat{L} \in G$, there is $\hat{L}^{-1}$ in G

s.t. $\hat{L} \circ \hat{L}^{-1} = \hat{L}^{-1} \circ \hat{L} = L$ the identity transformation.

$$\hat{L}(A) = S^{-1}AS.$$

$$\hat{L}^{-1}(A) = SAS^{-1}$$

$$\therefore \quad \hat{L}(\hat{L}^{-1}(A)) = S^{-1}(SAS^{-1})S$$

$$= A \quad \forall A$$

$$\hat{L}^{-1}(\hat{L}(A)) = S(S^{-1}AS)S^{-1}$$

$$= A \quad \forall A$$

$*$-congruence

- A matrix $A \in M_n$ is $*$-congruent to $B \in M_n$ if there is an invertible matrix S such that $A = S^*BS$.
- There is no easy canonical form under $*$-congruence for general matrix.[2]
- Every Hermitian matrix $A \in M_n$ is $*$-congruent to $I_p \oplus -I_q \oplus 0_{n-p-q}$. The triple $\nu(A) = (p, q, n-p-q)$ is known as the inertia of A.
- Two Hermitian matrices are $*$-congruent if and only if they have the same inertia.

Proof. Use the unitary congruence/similarity results. □

Congruence or t-congruence

- A matrix $A \in M_n$ is t-congruent to $B \in M_n$ if there is an invertible matrix S such that $A = S^tBS$.
- There is no easy canonical form under t-congruence for general matrices; see footnote 2.
- Every complex symmetric matrix $A \in M_n$ is t-congruent to $I_k \oplus 0_{n-k}$, where $k = \operatorname{rank}(A)$.
- Every skew-symmetric $A \in M_n$ is t-congruent to 0_{n-2k} and k copies of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
- The rank of a skew-symmetric matrix $A \in M_n$ is even. $A = -A^t$
- Two symmetric (skew-symmetric) matrices are t-congruent if and only if they have the same rank.

Proof. Use the unitary congruence results. □

[2] Roger A. Horn and Vladimir V. Sergeichuk, Canonical forms for complex matrix congruence and $*$-congruence, Linear Algebra Appl. (2006), 1010-1032.

Symmetric matrices

$$A = \begin{bmatrix} 1+i & 3 & 2-i \\ 3 & 2-i & 1 \\ 2-i & 1 & 3+i \end{bmatrix} \quad S^t A S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 1 $\quad \downarrow$ U unitary

$$U^t A U = \left[\begin{array}{cc|c} s_1 & 0 & \\ 0 & s_k & 0 \\ \hline 0 & & 0 \end{array}\right] \quad s_1 \geq \dots \geq s_k > 0$$

$$A = \cancel{U A U^*}$$

$$A = (U^t)^* \left[\begin{array}{c|c} s_1 \ \ddots \ s_k & 0 \\ \hline 0 & 0 \end{array}\right] U^*$$

$$A^* A = U \left[\begin{array}{c|c} s_1 \ \ddots \ s_k & 0 \\ \hline 0 & 0 \end{array}\right] U^t (U^t)^* \left[\begin{array}{c|c} s_1 \ \ddots \ s_k & 0 \\ \hline 0 & 0 \end{array}\right] U^*$$

$$= U \left[\begin{array}{c|c} s_1^2 \ \ddots \ s_k^2 & 0 \\ \hline 0 & 0 \end{array}\right] U^*$$

Step 2

$$\begin{bmatrix} \frac{1}{\sqrt{s_1}} & & & 0 \\ & \ddots & & \\ & & \frac{1}{\sqrt{s_k}} & \\ 0 & & & \ddots \end{bmatrix} U^t A U \begin{bmatrix} \frac{1}{\sqrt{s_1}} & & & 0 \\ & \ddots & & \\ & & \frac{1}{\sqrt{s_k}} & \\ 0 & & & 1 \ \ddots \ 1 \end{bmatrix} = \left[\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array}\right].$$

$|u^t A u|$ is max $\quad$ u unit vector

$v = e^{it} u \quad$ v unit vector

$$v^t A v = e^{it} u^t A e^{it} u$$
$$= (e^{i2t}) \underbrace{u^t A u}_{= s_1} > 0$$

$V = [v | \cdots]$ unitary

$V^t A V$ is symmetric. $\quad (V^t A V)^t = V^t A^t V = V^t A V.$

$$\begin{array}{c} \| \\ \left[\begin{array}{c|ccc} a_{11} & a_{12} & \cdots & a_{1n} \\ \hline a_{12} & & & \\ a_{13} & & & \\ \vdots & & & \end{array}\right] \end{array} = \left[\begin{array}{c|c} s_1 & 0 \\ \hline 0 & * \end{array}\right]$$

Unitary congruence

- A matrix $A \in M_n$ is unitarily congruent to $B \in M_n$ if there is a unitary matrix U such that $A = U^t BU$.
- There is no easy canonical form under unitary congruence for general matrices.
- Every complex symmetric matrix $A \in M_n$ is unitarily congruent to $\sum_{j=1}^{k} s_j E_{jj}$, where $s_1 \geq \cdots \geq s_k > 0$ are the nonzero singular values of A.
- Every skew-symmetric $A \in M_n$ is unitarily congruent to 0_{n-2k} and

$$\begin{pmatrix} 0 & s_j \\ -s_j & 0 \end{pmatrix}, \qquad j = 1, \ldots, k,$$

  where $s_1 \geq \cdots \geq s_k > 0$ are nonzero singular values of A.
- The singular values of a skew-symmetric matrix $A \in M_n$ occur in pairs.
- Two symmetric (skew-symmetric) matrices are unitarily congruent if and only if they have the same singular values.

Proof. Suppose $A \in M_n$ is symmetric. Let $\mathbf{x} \in \mathbb{C}^n$ be a unit vector so that $\mathbf{x}^t A\mathbf{x}$ is real and maximum, and let $U \in M_n$ be unitary with $\mathbf{x}$ as the first column. Show that $U^t AU = [s_1] \oplus A_1$. Then use induction.

Suppose $A \in M_n$ is skew-symmetric. Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ be orthonormal pairs such that $\mathbf{x}^t A\mathbf{y}$ is real and maximum, and $U \in M_n$ be unitary with $\mathbf{x}, \mathbf{y}$ as the first two columns. Show that $U^t AU = \begin{pmatrix} 0 & s_1 \\ -s_1 & 0 \end{pmatrix} \oplus A_1$. Then use induction. □

Skew-Symmetric Matrices

Example

$$A=\begin{bmatrix} 0 & 1 & 5i \\ -1 & 0 & 2i \\ -5i & -2i & 0 \end{bmatrix}$$ has rank 2

Example

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$-A^t=\begin{bmatrix} -a_{11} & -a_{21} & -a_{31} \\ -a_{12} & -a_{22} & -a_{32} \\ -a_{13} & -a_{23} & -a_{33} \end{bmatrix}$$

$$A=\begin{bmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{bmatrix}$$

Example

$$A=\begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$$

$$S^tAS=\left.\begin{bmatrix} \begin{matrix}0 & 1\\ -1 & 0\end{matrix} & 0 & \\ 0 & \begin{matrix}0 & 1\\ -1 & 0\end{matrix} & \\ & & \begin{matrix}0 & 1\\ -1 & 0\end{matrix} \end{bmatrix}\right\} k.$$

A has rank $2k$

$$(S^tAS)^t = S^tA^tS = S^t(-A)S = -S^tAS$$

In general. $A=-A^t$.

there is a unitary U s.t.

$$B = U^tAU = \begin{bmatrix} \begin{matrix}0 & s_1\\ -s_1 & 0\end{matrix} & 0 & 0 & \\ 0 & \begin{matrix}0 & s_2\\ -s_2 & 0\end{matrix} & 0 & \\ 0 & 0 & \begin{matrix}0 & s_k\\ -s_k & 0\end{matrix} & \\ & & & 0 \end{bmatrix}$$

$$S^t \quad S$$

$$S^t\begin{bmatrix} \frac{1}{\sqrt{s_1}} & & & & & & \\ & \frac{1}{\sqrt{s_1}} & & & & & \\ & & \frac{1}{\sqrt{s_2}} & & & & \\ & & & \frac{1}{\sqrt{s_2}} & & & \\ & & & & \ddots & & \\ & & & & \frac{1}{\sqrt{s_k}} & & \\ & & & & & \frac{1}{\sqrt{s_k}} & \\ & & & & & & I_{n-2k} \end{bmatrix} U^tAU \begin{bmatrix} \frac{1}{\sqrt{s_1}} & & & & & 0 \\ & \frac{1}{\sqrt{s_1}} & & & & \\ & & \frac{1}{\sqrt{s_2}} & & & \\ & & & \frac{1}{\sqrt{s_2}} & & \\ 0 & & & & \ddots & \\ & & & & & I_{n-2k} \end{bmatrix} = \begin{bmatrix} \begin{matrix} & 1\\ -1 & \end{matrix} & 0 & 0 \\ 0 & \begin{matrix} & 1\\ -1 & \end{matrix} & 0 \\ 0 & 0 & \begin{matrix} & 1\\ -1 & \end{matrix} \end{bmatrix}$$

$\max |u_1^t A u_2|$ ~~max~~ of for a pair of orthonormal vectors $\{u_1, u_2\}$.

Let $V = [u_1 | u_2 | \; * \;]$.

Then prove

$$V^t A V = \left[\begin{array}{cc|c} 0 & u_1^t A u_2 & 0 \\ u_1^t A u_2 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$