Math 408

Will post the Exam I paper by Friday 5:00 pm. (Take home).

Due next Friday noon.

No homework due next Friday.

$U^*U = I$ → orthogonal matrix

If U is real, $U^tU = I = UU^t$

2.6 Remarks on real matrices

Theorem 2.6.1 *If $\boxed{A \in M_n(\mathbb{R})}$, then A is orthogonally similar to block triangular matrix (A_{ij}) such that A_{jj} is either 1×1 or of the forms $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.* ✓

$\begin{bmatrix} \begin{smallmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{smallmatrix} & * & * \\ 0 & \ddots & \\ & & \end{bmatrix}$

- *The matrix is normal, i.e., $AA^t = A^tA$, if and only if all the off-diagonal blocks are zero.*
- *The matrix A orthogonal, i.e., $AA^t = I_n$, if and only if the 1×1 diagonal blocks have the form $[1]$ or $[-1]$, and the entries in the 2×2 diagonal blocks satisfy $a^2 + b^2 = 1$.*
- *The matrix A is* **symmetric**, *i.e., $A = A^t$, if and only if it is orthogonally similar to a real diagonal matrices.*
- *The matrix A is* **symmetric** *and satisfies $x^tAx \geq 0$ for all $x \in \mathbb{R}^n$ if and only if it is orthogonally similar to a nonnegative diagonal matrices.*

Remark 2.6.2 *Let $A \in M_n(\mathbb{R})$. Then $\boxed{A = S + K}$ where $S = (A + A^t)/2$ is symmetric and $K = (A - A^t)/2$ is skew-symmetric, i.e., $K^t = -K$.*

- *Note that $x^tKx = 0$ for all $x \in \mathbb{R}^n$.* $x^tKx = (x^tKx)^t = x^tK^tx = x^t(-K)x = -x^tKx \Rightarrow x^tKx = 0$
- *Clearly, $x^tAx \in \mathbb{R}$ for all real vectors $x \in \mathbb{R}^n$, and the condition does not imply that A is symmetric as in the complex Hermitian case.* (for complex matrix, $A = H + iG$? 0)
- *The matrix A satisfies $x^tAx \geq 0$ for all if and only if $(A + A^t)/2$ has only nonnegative eigenvalues. The condition does not automatically imply that A is symmetric as in the complex Hermitian case.*
- *Every skew-symmetric matrix $K \in M_n(\mathbb{R})$ is orthogonally similar to 0_{2k} and*

$$\begin{pmatrix} 0 & s_j \\ -s_j & 0 \end{pmatrix}, \qquad j = 1, \ldots, k,$$

where $s_1 \geq \cdots \geq s_k > 0$ are nonzero singular values of A.

General U^tAU form:

1

Induction:

$U^t A [U] = \begin{bmatrix} \lambda_1 & * \cdots * \\ 0 & A_1 \end{bmatrix}$

$\begin{bmatrix} \begin{smallmatrix} \lambda_1 & * \\ 0 & \ddots \end{smallmatrix} & * \\ 0 & A_k \end{bmatrix}$

2

Real Normal:

$A = U^* \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} U^*$

3

Real Symmetric.

If $A = A^*$ & A is real, then $A = A^t$.

If $A = A^t$ & A is real, then $A = A^t$.

$U^* \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} U$, $U^t \begin{bmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_n \end{bmatrix} U$

$A_k - \lambda I$:

$\mu, \bar{\mu}$,

$A(x+iy) = \mu(x+iy)$

3 Eigenvalues and singular values inequalities

We study inequalities relating the eigenvalues, diagonal elements, singular values of matrices in this chapter.

first k nonzero.

For a Hermitian matrix A, let $\lambda(A) = (\lambda_1(A), \ldots, \lambda_n(A))$ be the vector of eigenvalues of A with entries arranged in descending order. Also, we will denote by $s(A) = (s_1(A), \ldots, s_n(A))$ the singular values of a matrix $A \in M_{m,n}$. For two Hermitian matrices, we write $A \geq B$ if $A - B$ is positive semidefinite.

3.1 Diagonal entries and eigenvalues of a Hermitian matrix

The following classical result is useful.

$$\overline{x^*Ax} = \overline{(x^*Ax)}^t = (x^*Ax)^* = x^*A^*x = x^*Ax$$

Lemma 3.1.1 (Rayleigh principle) *Let $A \in M_n$ be Hermitian. Then for any unit vector $\mathbf{x} \in \mathbb{C}^n$,*

$$\lambda_1(A) \geq \mathbf{x}^* A \mathbf{x} \geq \lambda_n(A).$$

$$x^*Ax = x^*U \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_n \end{bmatrix} U^*x \qquad (\eta_1 \cdots \eta_n) \qquad \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

The equalities hold at unit eigenvectors corresponding to the largest and smallest eigenvalues of A, respectively.

Proof. Done in homework problem. □

$$e_j^t A e_j = a_{11} \quad a_{jj} = (a_{ij})$$

If we take $\mathbf{x} = e_j$, we see that every diagonal entry of a Hermitian matrix A lies between $\lambda_1(A)$ and $\lambda_n(A)$.

We can say more in the following. To do that we need the notion of **majorization** and **doubly stochastic matrices.**

A matrix $D = (d_{ij}) \in M_n$ is doubly stochastic if $d_{ij} \geq 0$ and all the row sums and column sums of D equal 1.

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. We say that $\mathbf{x}$ is weakly majorized by $\mathbf{y}$, denoted by $\mathbf{x} \prec_w \mathbf{y}$ if the sum of the k largest entries of $\mathbf{x}$ is not larger than that of $\mathbf{y}$ for $k = 1, \ldots, n$; in addition, if the sum of the entries of $\mathbf{x}$ and $\mathbf{y}$, we say that $\mathbf{x}$ is majorized by $\mathbf{y}$, denoted by $\mathbf{x} \prec \mathbf{y}$. We say that $\mathbf{x}$ is obtained from $\mathbf{y}$ be a pinching if $\mathbf{x}$ is obtained from $\mathbf{y}$ by changing (y_i, y_j) to $(y_i - \delta, y_j + \delta)$ for two of the entries $y_i > j_j$ of y and some $\delta \in (0, y_i - y_j)$.

Examples

$$\begin{bmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & 100 \end{bmatrix} \qquad A = A^* = \begin{bmatrix} 1 & * & * & * \\ * & 2 & * & * \\ * & * & 3 & * \\ * & * & * & 1000 \end{bmatrix} \Rightarrow \lambda_1(A) \geq 1000, \quad \lambda_n(A) \leq 1.$$

Can we construct a matrix $A = A^t$ real such that A has eigenvalues $3, 2, 1$, diagonal elements $(2, 2, 2)$

Example. D.S. matrices

$$\begin{bmatrix} 0.1 & 0.9 \\ 0.9 & 0.1 \end{bmatrix} \qquad n=3 \quad \begin{bmatrix} 0.1 & 0.2 & 0.7 \\ 0.7 & 0.1 & 0.2 \\ 0.2 & 0.7 & 0.1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$0 \leq d \leq 1 \qquad \begin{bmatrix} d & 1-d \\ 1-d & d \end{bmatrix}$$

DS matrices

Note: $A = (a_{ij}) \in M_n$ with $a_{ij} \geq 0$ is d.s. if and only if

$$A \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

and

$$[1 \cdots 1] A = [1 \cdots 1]$$

Note. D_1, D_2 are ds.

$\Rightarrow$ ~~D_1, D_2~~ $D_1 D_2$ is ds.

~~Proof~~ ① $D_1 D_2 = (z_{ij}) \Rightarrow z_{ij} \geq 0$

Let $e = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ & $D_1 D_2 e = D_1 e = e$

$e^t D_1 D_2 = e^t D_2 = e^t$

Examples of $x \prec_w y$.

$(3,2,1) \prec_w (6,0,0)$ | $(3,2,1) \prec (6,0,0)$

$(3,2,1) \not\prec_w (4, \tfrac{1}{2}, 0)$

$(3,2,1)\ {}_w\!\not\succ (4, \tfrac{1}{2}, 0)$

$(3,2,1) \prec_w (7,0,0)$ & $(3,2,1) \not\prec (7,0,0)$

$$(M, 0 \cdots, 0) \succ \left(\tfrac{M}{2}, \tfrac{M}{2}, 0 \cdots, 0\right) \succ \left(\tfrac{M}{n}, \cdots, \tfrac{M}{n}\right)$$

Theorem 3.1.2 *Let* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *with* $n \geq 2$. *The following conditions are equivalent.*

(a) $\mathbf{x} \prec \mathbf{y}$.

(b) *There are vectors* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ *with* $k < n$, $\mathbf{x}_1 = \mathbf{y}$, $\mathbf{x}_k = \mathbf{x}$, *such that each* $\mathbf{x}_j$ *is obtained from* $\mathbf{x}_{j-1}$ *by pinching two of its entries.*

(c) $\mathbf{x} = D\mathbf{y}$ *for some doubly stochastic matrix.*

(c) $\Rightarrow$ (a) Suppose $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} d_{ij} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$.

We assume $x_1 \geq \cdots \geq x_n$
$y_1 \geq \cdots \geq y_n$.

Otherwise we prove

$$\boxed{P\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}} = D \boxed{Q\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = P^t D Q \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

For any $k \in \{1, \cdots, n\}$.

Let $v_k = [\underbrace{1\,1\,1\,1\,1}_{k}\,0 \cdots 0]^t$.

Then $\sum_{i=1}^{k} x_i = v_k^t \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$= v_k^t D \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

$= [c_1 \, c_2 \cdots c_n] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

$= c_1 y_1 + \cdots + c_k y_k + c_{k+1} y_{k+1} + \cdots + c_n y_n$

$\leq c_1 y_1 + \cdots + c_k y_k + (c_{k+1} + \cdots + c_n) y_k$

$= c_1 y_1 + \cdots + c_k y_k + [(1 - c_1) y_k + \cdots + (1 - c_k) y_k]$

$\leq c_1 y_1 + \cdots + c_k y_k + (1 - c_1) y_1 + \cdots + (1 - c_k) y_k = y_1 + \cdots + y_k$.

$0 \leq c_i \leq 1$

$c_1 + \cdots + c_n$
$= v_k^t D \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$
$= v_k^t \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = k$.

$k - (c_1 + \cdots + c_k) = c_{k+1} + \cdots + c_n$

$(1 - c_1) \cdots (1 - c_k)$

$$e^t x = e^t D y = e^t y \;\therefore\; \sum^n x_i = \sum^n y_i$$

$$P\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \prec Q\begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix}$$

with $x = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $y = \begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix}$; $P x = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, $Q y = D\begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$

(a) $\Rightarrow$ (b). Consider $y = (y_1, \cdots, y_n)^t$

$x = (x_1, \cdots, x_n)^t$ . .

We prove the result by induction on $n \geq 2$.

If $n = 2$, $y_1 \geq x_1 \geq x_2 \geq y_2$

$(x_1, x_2) = (y_1 - d, y_2 + d)$

$d = y_1 - x_1$

Assume $n > 3$. Let k be the max integer so that

$y_k \geq x_1$.

Case 1° If $k = n$.,

then $y_n \geq x_1 \geq \cdots \geq x_n \geq y_n$. $\therefore$ $x_1 = \cdots = x_n$

and $\dfrac{y_1 + \cdots + y_n}{n} = \dfrac{x_1 + \cdots + x_n}{n} = x_n$

$\Rightarrow$ $y_1 = \cdots = y_n = x_n = \cdots = x_1$

Case 2° If $k < n$,

then change (y_k, y_{k+1}) to $(x_k, \underbrace{y_k + y_{k+1} - x_k}_{\hat{y}_{k+1}})$

Then we get.

$(x_1, \cdots, x_n)^t$ & $(y_1, \cdots, y_{k-1}, x_1, \hat{y}_{k+1}, y_{k+2}, \cdots, y_n)^t$

Apply induction $(x_2, \cdots, x_n)^t$, $(y_1, \cdots, y_{k-1}, \hat{y}_{k+1}, y_{k+2}, \cdots, y_n)^t$

Clearly, the sum of entries of the two vectors in $\mathbb{R}^{n-1}$ are the same.

Case 1° For $l \leq k$.

$x_2 + \cdots + x_l \leq y_1 + \cdots + y_l$

Case 2° For $k < l$

$x_2 + \cdots + x_l \leq y_1 + \cdots + y_{k-1} + \hat{y}_{k+1} + y_l = y_1 + \cdots + y_l$ $-(x_1)$

Theorem 3.1.3 *Let $\mathbf{d}, \mathbf{a} \in \mathbb{R}^n$. The following are equivalent.* order of entries does not matter.

(a) *There is a complex Hermitian (real symmetric) $A \in M_n$ with entries of* $\mathbf{a}$ *as eigenvalues and entries of* $\mathbf{d}$ *as diagonal entries.*

(b) *The vectors satisfy* $\mathbf{d} \prec \mathbf{a}$.

If $A = \begin{pmatrix} d_1 & & * \\ & \ddots & \\ * & & d_n \end{pmatrix}$ has e.v. $(a_1, \dots, a_n)$,

then
$$\overset{S}{\begin{bmatrix} 0 & & 1 \\ & ⋰ & \\ 1 & & 0 \end{bmatrix}} A \overset{S^{-1}}{\begin{bmatrix} 0 & & 1 \\ & ⋰ & \\ 1 & & 0 \end{bmatrix}} = \begin{bmatrix} d_n & & * \\ & \ddots & \\ * & & d_1 \end{bmatrix}$$

has e.v. $a_1, \dots, a_n$.

For any order $d_{i_1} \cdots d_{i_n}$

$$PAP^t = \begin{bmatrix} d_{i_1} & & * \\ & \ddots & \\ * & & d_{i_n} \end{bmatrix}$$

Remark If $x \prec y$.

then the sum of the k smallest entries of x is ~~$S - \sum_{i=1} x_i$~~ $\geq$ k smallest entries of y

$$S = \sum^n x_i = \sum^n y_i$$

$$\sum_{i=n-k+1}^{n} x_i = S - \sum_{i=1}^{n-k} x_i$$

$$\vee|$$

$$\sum_{i=n-k+1}^{n} y_i = S - \sum_{i=1}^{n-k} y_i$$