Math 408 Advanced Linear Algebra Examination 1 Due: noon, 7 October.

1. Let $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_n$ be such that $A \in M_k$ is invertible.

   (a) Show that $\begin{pmatrix} I_k & 0 \\ -CA^{-1} & I_{n-k} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I_k & -A^{-1}B \\ 0 & I_{n-k} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix}$, and
   $$\det(T) = \det(A)\det(D - CA^{-1}B).$$

   (b) Show that T is invertible if and only if $D - CA^{-1}B$ is invertible, and
   $$T^{-1} = \begin{pmatrix} I_k & -A^{-1}B \\ 0 & I_{n-k} \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & (D - CA^{-1}B)^{-1} \end{pmatrix} \begin{pmatrix} I_k & 0 \\ -CA^{-1} & I_{n-k} \end{pmatrix}.$$

2. Let $A \in M_7$ with minimal polynomial $(x-1)^2(x-i)^3$. Determine all possible Jordan forms of A.

3. (a) Let $A = J_k(\lambda)$ with $k \geq 2$ and $\lambda = \mu^2 \neq 0$. Show that there is an invertible matrix $S \in M_k$ such that $S^{-1}AS = \lambda I_k + 2\mu N_k + N_k^2 = (\mu I_k + N_k)^2$, and hence $A = B^2$ for $B = S(\mu I_k + N_k)S^{-1}$.

   $\begin{bmatrix} \lambda & 2\mu & 1 \\ 0 & \ddots & 2\mu \end{bmatrix}$ $\begin{bmatrix} \mu & 1 \\ & \ddots \end{bmatrix} \in M_k$. $\searrow J_k(\mu)$

   (b) Show that for every invertible matrix $T \in M_n$, there is $R \in M_n$ such that $T = R^2$.

4. Suppose $B = J_n(0)$. Then B^2 has Jordan form
   $$\begin{cases} J_k(0) \oplus J_k(0) & \text{if } n = 2k, \\ J_k(0) \oplus J_{k+1}(0) & \text{if } n = 2k+1. \end{cases}$$

   $T = S^{-1}[\ \ ]S^{-1}$

   Deduce that there is no matrix $T \in M_6$ such that $T^2 = J_2(0) \oplus J_4(0)$.

5. Show that $A, B \in M_2$ are unitary if and only if $tr\,A = tr\,B$, $tr\,A^2 = tr\,B^2$, $tr\,AA^* = tr\,BB^*$.

   Hint: To prove the sufficiency, first show that A and B have the same eigenvalues, and hence by unitary similarities, the triangular forms T_1 and T_2 of A and B have the same diagonal entries. Then show that the $(1,2)$ entries of the two triangular matrices can be changed to the same nonnegative number by applying diagonal unitary similarities $D_1^*T_1D_1$ and $D_2^*T_2D_2$.

6. Let $w = e^{i2\pi/3}$, and $F = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w \end{pmatrix}$.

   w w^2

   (a) Show that F is unitary. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

   (b) Suppose $P = E_{12} + E_{23} + E_{31} \in M_3$. Show that the $PF = FD$ with $D = \text{diag}\,(1, w, w^2)$.

   (c) Suppose $A = a_1I + a_2P + a_3P^2 = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_3 & a_1 & a_2 \\ a_2 & a_3 & a_1 \end{pmatrix}$. Determine F^*AF.

7. Suppose $B \in M_n$ is invertible. Show that $B = LV$ for some unitary V and lower triangular matrix L in M_n. Deduce that every invertible positive semidefinite matrix A can be written as $L_1L_1^*$ such that L_1 is lower triangular.

8. (a) Show that $A \in M_n$ is normal if and only if $\|Av\| = \|A^*v\|$ for all $v \in \mathbb{C}^n$.

   (b) Show that $A \in M_n$ is skew-symmetric if and only if $u^tAu = 0$ for all $u \in \mathbb{C}^n$.

$(M, 0 \cdots, 0) \to (\frac{M}{2}, \frac{M}{2}, 0 \cdots, 0)$

$\to (\frac{M}{n}, \cdots, \frac{M}{n})$

Theorem 3.1.2 *Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ with $n \geq 2$. The following conditions are equivalent.*

(a) $\mathbf{x} \prec \mathbf{y}$.

(b) *There are vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ with $k < n$, $\mathbf{x}_1 = \mathbf{y}$, $\mathbf{x}_k = \mathbf{x}$, such that each $\mathbf{x}_j$ is obtained from $\mathbf{x}_{j-1}$ by pinching two of its entries.*

(c) $\mathbf{x} = D\mathbf{y}$ *for some doubly stochastic matrix.*

(c) $\Rightarrow$ (a) Suppose $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (d_{ij}) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

We assume $x_1 \geq \cdots \geq x_n$

$y_1 \geq \cdots \geq y_n$.

For any $k \in \{1, \cdots, n\}$.

Let $v_k = [\underbrace{1\,1\,1\,1\,1}_{k}\,0 \cdots 0]^t$.

Then $\sum_{i=1}^{k} x_i = v_k^t \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$= v_k^t D \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

$= [c_1\, c_2 \cdots c_n] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

$= c_1 y_1 + \cdots + c_k y_k + c_{k+1} y_{k+1} + \cdots + c_n y_n$

$\leq c_1 y_1 + \cdots + c_k y_k + (c_{k+1} + \cdots + c_n) y_k$

$= c_1 y_1 + \cdots + c_k y_k + [(1-c_1) y_k + \cdots + (1-c_k) y_k]$

$\leq c_1 y_1 + \cdots + c_k y_k + (1-c_1) y_1 + \cdots + (1-c_k) y_k = y_1 + \cdots + y_k$.

$0 \leq c_i \leq 1$

$c_1 + \cdots + c_n$

$= v_k^t D \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

$= v_k^t \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = k$.

$k - (c_1 + \cdots + c_k) = c_{k+1} + \cdots + c_n$

$(1-c_1) \cdots (1-c_k)$

$e^t x = e^t D y = e^t y$. $\therefore \sum^n x_i = \sum y_i$

Otherwise we prove

$\boxed{P\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}} = D \boxed{Q\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}}$

$\Rightarrow \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = P^t D Q \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$x \qquad y$

$P\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \prec Q\begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \quad D\begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$

Objective: Relations between the eigenvalues $\lambda_1, \dots, \lambda_n$ of $\left\{ \begin{array}{l} A = A^t \in M_n(\mathbb{R}) \\ A = A^* \in M_n \end{array} \right.$

and diagonal elements $a_1, \dots, a_n$ of A.

Answ. $(a_1, \dots, a_n) \prec (\lambda_1, \dots, \lambda_n)$

$$\sum^n \lambda_i = \sum^n a_i$$

Sum of k largest entries in $(\lambda_1 \dots \lambda_n)$

$\geq$ Sum of k largest entries in $(a_1, \dots, a_n)$

Example. For $(\lambda_1, \lambda_2, \lambda_3) = (6, 2, 1)$, there is a real symmetric

$$A = \begin{bmatrix} 3 & * & * \\ * & 3 & * \\ * & * & 3 \end{bmatrix}$$ with eigenvalues $6, 2, 1$.

Done: $(c) \Rightarrow (a) \Rightarrow (b)$ ✓✓✓

Finish the proof of $(b) \Rightarrow (c)$.

Suppose $y = x_1, x_2, \dots, x_k = x$.

such that x_j is obtained from x_{j-1} by "pinching" to entries, i.e., change to entries

$\alpha_p > \alpha_q$ to $\alpha_p - \varepsilon, \alpha_q + \varepsilon$, $\varepsilon \in (0, \alpha_p - \alpha_q)$.

Note that

$$\begin{bmatrix} t & (1-t) \\ 1-t & t \end{bmatrix} \begin{bmatrix} \alpha_p \\ \alpha_q \end{bmatrix} = \begin{bmatrix} \alpha_p - \varepsilon \\ \alpha_q + \varepsilon \end{bmatrix}$$

with $t\alpha_p + (1-t)\alpha_q = \alpha_p - \varepsilon$. $\Rightarrow (\alpha_p - \alpha_q)t = \alpha_p - \varepsilon - \alpha_q$

$$t = \frac{\alpha_p - \varepsilon - \alpha_q}{\alpha_p - \alpha_q} > 0.$$

So let T_j be doubly stochastic obtained by changing the 2×2 submatrix of the identity matrix at rows/columns p & q, by $\begin{bmatrix} t & 1-t \\ 1-t & t \end{bmatrix}$.

Example.

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & t & & 1-t \\ & & & 1 & \\ & & 1-t & & t \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \hat{\alpha}_3 \\ \alpha_4 \\ \hat{\alpha}_5 \end{bmatrix}$$

$\hat{\alpha}_3 \to t\alpha_3 + (1-t)\alpha_5$

$\hat{\alpha}_5 \to (1-t)\alpha_3 + t\alpha_5$

So $y = x_1, \ldots, x_k = x$

$T_{k-1} \cdots T_2 T_1 y = x$

Note that T_i's are doubly stochastic $\therefore$

$D = T_{k-1} \cdots T_1$ is doubly stochastic.

Example:

$$y = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix} \qquad x = \begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix}$$

$= x_1$

$\succ$

$$x_2 = \begin{bmatrix} 5 \\ 5 \\ 1 \end{bmatrix} \succ$$

$$T_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

$\therefore D = T_2 T_1$ is d.s. s.t.

$Dy = x$.

~~7+5~~

$t \cdot 7 + (1-t) \cdot 3 = 5$, $t = \frac{1}{2}$

$(1-t)7 + t\,3 = 5$

$t \cdot 5 + (1-t) \cdot 1 = 4$

$5t - t = 4 - 1 = 3$

$t = \frac{3}{4}$

(a) $\Rightarrow$ (b). Consider $y = (y_1, \dots, y_n)^t$

$x = (x_1, \dots, x_n)^t$ . .

We prove the result by induction on $n \geq 2$.

If $n = 2$, $y_1 \geq x_1 \geq x_2 \geq y_2$

$(x_1, x_2) = (y_1 - d, y_2 + d)$

$d = y_1 - x_1$

Assume $n > 3$. Let k be the max integer so that

$y_k \geq x_1$.

Case 1° If $k = n$,

then $y_n \geq x_1 \geq \dots \geq x_n \geq y_n$. $\therefore$ $x_1 = \dots = x_n$

and $\frac{y_1 + \dots + y_n}{n} = \frac{x_1 + \dots + x_n}{n} = x_n$

$\Rightarrow$ $y_1 = \dots = y_n = x_n = \dots = x_1$

Case 2° If $k < n$,

then change (y_k, y_{k+1}) to $(x_k, y_k + y_{k+1} - x_k)$

$\hat{y}_{k+1}$

Then we get

$(x_1, \dots, x_n)^t$ & $(y_1, \dots, y_{k-1}, x, \hat{y}_{k+1}, y_{k+2}, \dots, y_n)^t$

Apply induction $(x_2, \dots, x_n)^t$, $(y_1, \dots, y_{k-1}, \hat{y}_{k+1}, y_{k+2}, \dots, y_n)^t$

Clearly, the sum of entries of the two vectors in $\mathbb{R}^{n-1}$ are the same

Case 1° For $l \leq k$.

$x_2 + \dots + x_l \leq y_1 + \dots + y_l$

Case 2° For $k < l$

$x_2 + \dots + x_l \leq y_1 + \dots + y_{k-1} + \hat{y}_{k+1} + y_l = y_1 + \dots + y_l$

Theorem 3.1.3 *Let $\mathbf{d}, \mathbf{a} \in \mathbb{R}^n$. The following are equivalent.* order of entries does not matter.

(a) *There is a complex Hermitian (real symmetric) $A \in M_n$ with entries of $\mathbf{a}$ as eigenvalues and entries of $\mathbf{d}$ as diagonal entries.*

(b) *The vectors satisfy $\mathbf{d} \prec \mathbf{a}$.*

(a)⇒(b). Assume

$$U^*\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}U = \begin{pmatrix} d_1 & & * \\ & \ddots & \\ * & & d_n \end{pmatrix}, \quad U \text{ unitary.}$$

If $U = (u_{ij})$,

$$\begin{pmatrix} \bar u_{11} & \bar u_{21} & \cdots & \bar u_{n1} \\ \bar u_{12} & & & \vdots \\ \bar u_{1n} & \bar u_{2n} & \cdots & \bar u_{nn} \end{pmatrix}\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}\begin{pmatrix} u_{11} & \cdots & u_{1n} \\ \vdots & & \vdots \\ u_{n1} & \cdots & u_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} d_1 & & * \\ & \ddots & \\ * & & d_n \end{pmatrix}$$

$$d_1 = (\bar u_{11} \cdots \bar u_{n1})\begin{pmatrix} a_1 u_{11} \\ a_2 u_{21} \\ \vdots \\ a_n u_{n1} \end{pmatrix} = a_1|u_{11}|^2 + a_2|u_{21}|^2 + \cdots + |u_{n1}|^2$$

$$d_i = (\bar u_{1i} \cdots \bar u_{ni})\begin{pmatrix} a_1 u_{1i} \\ u_{2i} \\ \vdots \\ a_n u_{ni} \end{pmatrix} \quad \forall i$$

$$\begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} |u_{11}|^2 & |u_{21}|^2 & \cdots & |u_{n1}|^2 \\ |u_{12}|^2 & |u_{22}|^2 & \cdots & |u_{n2}|^2 \\ & \vdots & & \end{pmatrix}\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

$$\det \quad D = \begin{pmatrix} |u_{11}|^2 & \cdots & |u_{n1}|^2 \\ \vdots & & \vdots \\ |u_{1n}|^2 & \cdots & |u_{nn}|^2 \end{pmatrix}$$

Then D is nonnegative

& the ith row (col.) sum $|u_{1i}|^2 + \cdots + |u_{ni}|^2 = 1$

because the (i,i) entry of $U^*U = I$, ~~UU^*~~ I_n

If $A = \begin{pmatrix} d_1 & & * \\ & \ddots & \\ * & & d_n \end{pmatrix}$ has e.v. $(a_1, \dots, a_n)$,

then

$$\overset{S}{\begin{bmatrix} 0 & & 1 \\ & ⋰ & \\ 1 & & 0 \end{bmatrix}} A \overset{S^{-1}}{\begin{bmatrix} 0 & & 1 \\ & ⋰ & \\ 1 & & 0 \end{bmatrix}} = \begin{bmatrix} d_n & & * \\ & \ddots & \\ * & & d_1 \end{bmatrix}$$

has e.v. $a_1, \dots, a_n$.

For any order $d_{i_1} \cdots d_{i_n}$

$$PAP^t = \begin{bmatrix} d_{i_1} & & * \\ & \ddots & \\ * & & d_{i_n} \end{bmatrix}$$

Remark If $x \prec y$, then the sum of the k smallest entries of x is ~~$\sum_{i=1}^{k} x_i$~~ k smallest entries of y

$$s = \sum^n x_i = \sum^n y_i$$

$$\sum_{i=n-k+1}^{n} x_i = s - \sum_{i=1}^{n-k} x_i$$

$$\sum_{i=n-k+1}^{n} y_i = s - \sum_{i=1}^{n-k} y_i$$

Similarly, the ith column sum of D is 1 because it is the (i,i) entry of $UU^* = I_n$.

(b) $\Rightarrow$ (a) Assume $(d_1, \dots, d_n) \prec (a_1, \dots, a_n)$.

We prove by induction on n that there is a real symmetric $A \in M_n(\mathbb{R})$ s.t. A has diagonal entries $d_1, \dots, d_n$ & eigenvalues $a_1, \dots, a_n$.

$n = 2$ Let $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} d_1 & * \\ * & d_2 \end{bmatrix}$

$d_1 = a_1 \cos^2\theta + a_2 \sin^2\theta$. $\quad d_2 = a_1 \sin^2\theta + a_2 \cos^2\theta$.

want to find $\underset{= \cos^2\theta}{t} \in [0,1]$ s.t.

$$t a_1 + (1-t) a_2 = d_1, \text{ so}$$

$$t(a_1 - a_2) = d_1 - a_2$$

$$t = \frac{d_1 - a_2}{a_1 - a_2}.$$

Note that if $a_1 = a_2$ then $d_1 = d_2$. We can let $A = a_1 I_2$.

Suppose $n \geq 3$, & the results hold for matrices of sizes $< n$.

~~Choose~~ Assume $a_1 \geq \dots \geq a_n$

$d_1 \geq \dots \geq d_n$

Choose the maximum k s.t.

$a_k \geq d_1$.

Case 1' If $k = n$, $a_n \geq d_1 \geq \dots \geq d_n \geq a_n$

& $\sum d_i = \sum a_i \Rightarrow a_1 = \dots = a_n$ & $d_1 = \dots = d_n$ $\Big\}$ So $A = a_1 I_n$ is the required matrix

Case 2°

Assume $k < n$.

Then construct

$$\begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}a_k & 0 \\ 0 & a_{k+1}\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta \\ +\sin\theta & \cos\theta\end{bmatrix} = \begin{bmatrix}d_1 & * \\ * & \underbrace{\ast}_{a_k+a_{k+1}-d_1}\end{bmatrix}$$

~~Consider~~ Claim:

$$\underline{(a_k + a_{k+1} - d_1, a_1, \dots, a_{k-1}, a_{k+2}, \dots, a_n)}$$

$$\succ \underline{(d_2, \dots, d_n)}.$$

If the claim holds. then

$$\left[\begin{array}{c|c}\begin{matrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{matrix} & 0 \\ \hline 0 & I_{n-2}\end{array}\right]\left[\begin{array}{c|c}\begin{matrix}a_k & 0 \\ 0 & a_{k+1}\end{matrix} & 0 \\ \hline 0 & \begin{matrix}a_1 & & & 0 \\ & a_{k-1} & & \\ & & a_{k+2} & \\ 0 & & & a_n\end{matrix}\end{array}\right]\left[\begin{array}{c|c}\begin{matrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{matrix} & 0 \\ \hline 0 & I_{n-2}\end{array}\right] = \left[\begin{array}{c|c|c}d_1 & * & \\ \hline * & a_k+a_{k+1}-d_1 & 0 \\ \hline & 0 & \begin{matrix}a_1 & & 0 \\ & a_{k-1} & \\ 0 & & a_{k+2} \\ & & & a_n\end{matrix}\end{array}\right]$$

By induction, there is $Q \in M_{n-1}(\mathbb{R})$ orthogonal s.t.

$$Q^t \left[\begin{array}{c|c}a_k+a_{k+1}-d_1 & 0 \\ \hline 0 & \begin{matrix}a_1 & & & \\ & a_{k-1} & & \\ & & a_{k+1} & \\ & & & a_n\end{matrix}\end{array}\right] Q = \begin{bmatrix}d_2 & & * \\ & \ddots & \\ * & & d_n\end{bmatrix},$$

Then

$$\left[\begin{array}{c|c}1 & \\ \hline & Q^t\end{array}\right]\left[\begin{array}{c|c}\begin{matrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{matrix} & \\ \hline & I_{n-2}\end{array}\right]\left[\begin{array}{c|c}\begin{matrix}a_k & \\ & a_{k+1}\end{matrix} & \\ \hline & \begin{matrix}a_1 & & & \\ & a_{k-1} & & \\ & & a_{k+2} & \\ & & & a_n\end{matrix}\end{array}\right]\left[\begin{array}{c|c}\begin{matrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{matrix} & \\ \hline & I_{n-2}\end{array}\right]\left[\begin{array}{c|c}1 & \\ \hline & Q\end{array}\right]$$

$$= \begin{bmatrix}d_1 & & & * \\ & d_2 & & \\ & & \ddots & \\ * & & & d_n\end{bmatrix}$$

Finally, prove

$$(\overbrace{d_2, \dots, d_n}^{n-1}) \prec (\overbrace{a_k + a_{k+1} - d_1, a_1, \dots, a_{k-1}, a_{k+1}, \dots a_n}^{n-1}).$$

$$(a_1 \geq \dots \geq a_k \geq a_{k+1} \geq \dots a_n)$$

Consider l:

Case 1° For $l \leq k$.

$$d_2 + \dots + d_l \leq a_1 + \dots + a_{l-1}$$

which is true because

$$a_1 + \dots + a_{l-1} \geq d_1 + \dots + d_{l-1} \geq d_2 + \dots + d_l$$

Case 2° For $l > k$.

$$d_2 + \dots + d_l \leq a_1 + \dots + a_{k-1} + (a_k + a_{k+1} - d_1) + \dots + a_l$$

which is true because

$$d_1 + \dots + d_l \leq a_1 + \dots + a_{k-1} + a_l \quad ✓$$