Focus of Previous lecture:

$A = A^t \in M_n(\mathbb{R})$, ~~Complex Her~~ $A = A^* \in M_n$

with prescribed eigenvalues $a_1 \geq \cdots \geq a_n$

diagonal entries $d_1 \geq \quad \geq d_n$

$\Leftrightarrow \quad (d_1, \ldots, d_n) \prec (a_1, \ldots, a_n)$

Now we consider other techniques:

Recall $\quad V \subseteq \mathbb{C}^n$ is a subspace means

(1) $v_1, v_2 \in V \Rightarrow v_1 + v_2 \in V$.

(2) $\lambda \in \mathbb{C}, v \in V \Rightarrow \lambda v \in V$.

$Col(A)$

$Row(A)$

$Null(A)$.

In general a basis for V is a linearly independent set $\mathcal{B}$ of V so that every vector in V is a linear combination of the vectors in the set $\mathcal{B}$.

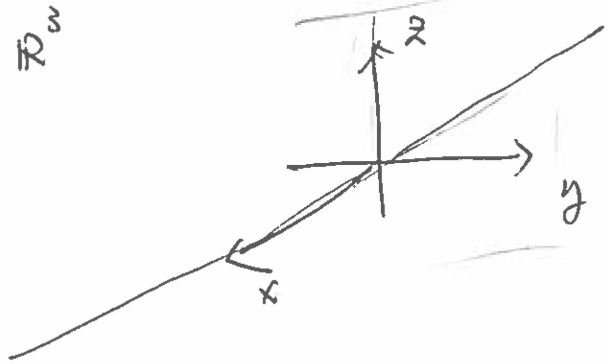
The dimension of V is the number of vectors in $\mathcal{B}$

3.2 Max-Min and Min-Max characterization of eigenvalues

In this subsection, we give a Max-Min and Min-Max characterization of eigenvalues of a Hermitian matrix.

Lemma 3.2.1 *Let* V_1 *and* V_2 *be subspaces of* $\mathbb{C}^n$ *such that* $\dim(V_1) + \dim(V_2) > n$, *then* $V_1 \cap V_2 \neq \{0\}$.

Example



$V_1 = \left\{ \begin{bmatrix} \lambda \\ 0 \\ 0 \end{bmatrix} : \lambda \in \mathbb{R} \right\}$

dim 1 Basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

$V_2 = \left\{ \begin{bmatrix} 0 \\ \lambda \\ \mu \end{bmatrix} : \lambda, \mu \in \mathbb{R} \right\}$

Basis $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Proof: Let $B_1 = \{u_1, \dots, u_k\}$ be a basis for V_1, $\dim V_1 = k$

$B_2 = \{v_1, \dots, v_m\}$ be a basis for V_2, $\dim V_2 = m$.

$m + k > n$.

Want to show there are $a_1, \dots, a_k, b_1, \dots, b_m \in \mathbb{C}$ not all zeros such that

$$\mathbb{C}^n \ni 0 = a_1 u_1 + \dots + a_k u_k - (b_1 v_1 + \dots + b_m v_m)$$

i.e.

$$\mathbb{C}^n \ni 0 = [\underbrace{u_1 | u_2 \cdots | u_k}_{k} | \underbrace{v_1 | \cdots | v_k}_{m}] \left.\begin{bmatrix} a_1 \\ \vdots \\ a_k \\ -b_1 \\ \vdots \\ -b_m \end{bmatrix}\right\} k+m$$

$$\boxed{Ax = 0}$$

Because $m+k>n$, in the row echelon form

$$[A|0] \to [\ \ \ 0\ ] \qquad A = [u_1 \cdots u_k\ v_1 \cdots v_k]$$

will have non-leading ones. So there is a non-trivial solution. Say $a_1, \dots, a_k$ are not all zero. Then

$$0 \neq a_1 u_1 + \dots + a_k u_k = b_1 v_1 + \dots + b_m v_m.$$

Then

$$\lambda_k(X^*AX) \le [\bar a_1 \cdots \bar a_k] \overbrace{X^*AX}^{k\times k} \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix}$$

$$= [\bar b_k \cdots \bar b_n]\, U^* A \,[u_k \cdots u_n] \begin{bmatrix} b_k \\ \vdots \\ b_n \end{bmatrix}$$

$$= [\bar b_k \cdots \bar b_n] \begin{bmatrix} \lambda_k & & & 0 \\ & \lambda_{k+1} & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \begin{bmatrix} b_k \\ \vdots \\ b_n \end{bmatrix}$$

$$\le \lambda_k(A). \qquad \square$$

Thus $\lambda_k(A) = \max\{\lambda_k(X^*AX) : X \in M_{n,k},\ X^*X = I_k\}$

$A = A^*$ $\quad \lambda(A) = (\lambda_1(A), \dots, \lambda_n(A))$

$\lambda_1(A) \geq \dots \geq \lambda_n(A)$.

Theorem 3.2.2 *Let $A \in M_n$ be Hermitian. Then for $1 \leq k \leq n$,*

$$\begin{aligned}\lambda_k(A) &= \max\{\lambda_k(X^*AX) : X \in M_{n,k}, X^*X = I_k\} \\ &= \min\{\lambda_1(Y^*AY) : Y \in \cancel{\ \ }\ M_{n,n-k+1}, Y^*Y = I_{n-k+1}\}.\end{aligned}$$

Note:

$\underset{k\times n}{X^*}\underset{n\times n}{A}\underset{n\times k}{X} \in M_k$, is Hermitian $\quad (X^*AX)^* = X^*A^*(X^*)^* = X^*AX$ has eigenvalues $\mu_1 \geq \dots \geq \mu_k$.

Example: $X = \begin{bmatrix} 1&0&0\\0&1&0\\0&0&1\\0&0&0\\0&0&0\end{bmatrix}$ $\quad n=5,\ k=3$

$\begin{bmatrix} I_3 & 0_{3\times 2}\end{bmatrix}\overset{3\times 3}{\begin{bmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}}\begin{bmatrix} I_3 \\ 0_{2\times 3}\end{bmatrix}$ $\quad X^* \overset{5\times5}{\begin{bmatrix} A \end{bmatrix}} \overset{X}{\begin{bmatrix} 1&0&0\\0&1&0\\0&0&1\\0&0&0\\0&0&0\end{bmatrix}} = 3\overset{3}{\begin{bmatrix} A_{11}\end{bmatrix}}$

$\lambda_3(A_{11}) \leq \lambda_3(A)$

$A = A^*$ $\quad 5\times 5$

$\begin{bmatrix} & \end{bmatrix} \begin{array}{c}1\\3\\5\end{array}\begin{bmatrix} A \end{bmatrix}\begin{bmatrix} 1&0&0\\0&0&0\\0&1&0\\0&0&0\\0&0&1\end{bmatrix}$

① We show $\lambda_k(A) \leq \max\{\ \ \}$.

~~Choose~~ Assume $U^*AU = \begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}$ $\quad \lambda_i(A) = \lambda_i$.

Let $u_1 \dots u_k$ be the first k columns of U & $X = (u_1 - u_k)$.

$AX = X\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_k\end{bmatrix}$.

$X^*AX = X^*X\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_k\end{bmatrix} = I_k\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_k\end{bmatrix} = \begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_k\end{bmatrix}$.

$\lambda_k(A) = \lambda_k(X^*AX) \leq \max\{\lambda_k(Y^*AY) : Y \in M_{n,k}, Y^*Y = I_k\}$.

Next. we prove $\lambda_k(A) \geq \lambda_k(X^*AX)$ for all $X \in M_{n,k}$, with $X^*X = I_k$

Suppose $X \in M_{n,k}$ with $X^*X = I_k$.

and has columns $x_1, \ldots, x_k$.

Recall $U^*AU = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$. & U has columns $u_1 \ldots u_n$.

Consider $V_1 = \mathrm{Span}\{\underline{x_1, \ldots, x_k}\}$, has dim k.

$V_2 = \mathrm{Span}\{u_k, \ldots, u_n\}$ has dim $n-k+1$

so that $\dim V_1 + \dim V_2 = n+1$

Then there is $v \neq 0$

s.t. $v = a_1 x_1 + \cdots + a_k x_k = b_k u_k + \cdots + b_n u_n$ $(*)$

We may assume $\|v\| = 1$. by dividing $(*)$ by $\|v\|$

Note

$$v = \overbrace{[x_1 | \cdots | x_k]}^{X} \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix} = [u_k \cdots u_n] \begin{bmatrix} b_k \\ \vdots \\ b_n \end{bmatrix}$$

&

$$1 = v^*v = [\bar{a}_1 \cdots \bar{a}_k] \underbrace{X^*X}_{I_k} \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix} = \Sigma |a_k|^2$$

$$1 = v^*v = [\bar{b}_k \cdots \bar{b}_n] [u_k \cdots u_n]^* [u_k \cdots u_n] \begin{bmatrix} b_k \\ \vdots \\ b_n \end{bmatrix}$$

$$= \Sigma |b_i|^2 .$$

3.3 Change of eigenvalues under perturbation

(real symmetric)

Theorem 3.3.1 *Suppose $A, B \in M_n$ are Hermitian such that $A \geq B$. Then $\lambda_k(A) \geq \lambda_k(B)$ for all $k = 1, \ldots, n$.*

$A - B$ is positive semidefinite.

Note: A & B may have ~~neg~~ positive / negative values. $A-B$ has positive eigenvalues

$\Rightarrow$ $\lambda(A) - \lambda(B)$ is non-negative vectors.

Proof: Let $A = B + P$ where P is psd

Suppose $\lambda_k(X^*BX) = \lambda_k(B)$ for $X \in M_{n,k}$ with $X^*X = I_k$.

and Choose u so that ~~$X^*Bu = \lambda_k(B)$ where $(X^*BX)u = \lambda_k(B)u$~~

$\boxed{X^*AXu = \cancel{\lambda_k(A)u}\ \lambda_k(X^*AX)u}$

$\|u\| = 1$.

Then $\lambda_k(B) = \lambda_k(X^*BX)$ ~~$\lambda_k(B)$~~ $\leq u^*(X^*BX)u$

$\leq u^*(X^*BX)u + u^*X^*P(Xu)$

1×k k×n n×n n×k k×1

y unit vector

$y^*Py \geq 0$

$\lambda_n(P) \geq 0$

$= u^*(X^*(B+P)X)u$

$\leq$ ~~[illegible] $\lambda_k(A)$~~.

$\leq \lambda_k(X^*AX)$

$= \lambda_k(A)$

Lemma 3.3.3 *Suppose $A \in M_{m,n}$ has nonzero singular values $s_1 \geq \cdots \geq s_k$. Then $\begin{pmatrix} 0_m & A \\ A^* & 0_n \end{pmatrix}$ has nonzero eigenvalues $\pm s_1, \ldots, \pm s_k$.*

Theorem 3.3.4 *Let $A, B, C \in M_{m,n}$ with singular values $a_1 \geq \cdots \geq a_n, b_1 \geq \cdots \geq b_n$ and $c_1, \ldots, c_n$, respectively. Then*

$$\sum_{j=1}^{k} (c_{r_j} - a_{r_j}) \leq \sum_{j=1}^{k} b_j.$$

Proof: Suppose A is $m \times n$ with nonzero singular values $s_1 \geq \cdots \geq s_k$. Then there are unitary $U \in M_m$, $V \in M_n$ s.t.

$$U^*AV = \begin{array}{c} \left[\begin{array}{c|c} \begin{matrix} s_1 & & \\ & \ddots & \\ & & s_k \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right] \begin{matrix} \\ \} m-k \end{matrix} \\ \qquad\quad n-k \end{array}$$

Consider

$$\begin{bmatrix} U^* & 0 \\ 0 & V^* \end{bmatrix} \begin{bmatrix} 0_m & A \\ A^* & 0_n \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}$$

$$= \begin{bmatrix} 0_m & U^*AV \\ \underline{V^*A^*U} & 0_n \end{bmatrix}$$

$$= \begin{array}{c} \qquad\qquad\qquad\qquad\qquad\quad n-k \\ \left[\begin{array}{c|c|c|c} 0_k & 0 & \begin{matrix} s_1 & & \\ & \ddots & \\ & & s_k \end{matrix} & 0 \\ \hline 0 & 0_{m-k} & 0 & 0 \\ \hline \begin{matrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_k \end{matrix} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \begin{matrix} \\ \} m-k \\ \\ \\ \end{matrix} \end{array}$$

$$= P^t \left[\begin{array}{c|c} \begin{matrix} \begin{matrix} & s_1 \\ s_1 & \end{matrix} & & & \\ & \begin{matrix} & s_2 \\ s_2 & \end{matrix} & & \\ & & \ddots & \\ & & & s_k \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right] P$$