3.3 Change of eigenvalues under perturbation

$A = B + P \quad P$ is psd.

Theorem 3.3.1 *Suppose $A, B \in M_n$ are Hermitian such that $A \geq B$. Then $\lambda_k(A) \geq \lambda_k(B)$ for all $k = 1, \dots, n$.*

Proof. Let $A = B + P$, where P is positive semidefinite. Suppose $k \in \{1, \dots, n\}$. There is $Y \in M_{n,k}$ with $Y^*Y = I_k$ such that

$$\lambda_k(B) = \lambda_k(Y^*BY) = \max\{\lambda_k(X^*BX) : X \in M_{m,n}, X^*X = I_k\}.$$

Let $\mathbf{y} \in \mathbb{C}^k$ be a unit eigenvector of Y^*AY corresponding to $\lambda_k(X^*AX)$. Then

$$\begin{aligned}
\lambda_k(A) &= \max\{\lambda_k(X^*AX) : X \in M_{m,n}, X^*X = I_k\} \\
&\geq \lambda_k(Y^*AY) = \mathbf{y}^*Y^*(B+P)Y\mathbf{y} = \mathbf{y}^*Y^*BY\mathbf{y} + \mathbf{y}^*Y^*PY\mathbf{y} \\
&\geq \mathbf{y}^*Y^*BY\mathbf{y} \geq \lambda_k(Y^*BY) = \lambda_k(B).
\end{aligned}$$

$\square$

Theorem 3.3.2 (Lidskii) *Let $A, B, C = A + B \in M_n$ be Hermitian matrices with eigenvalues $a_1 \geq \cdots \geq a_n$, $b_1 \geq \cdots \geq b_n$, $c_1 \geq \cdots \geq c_n$, respectively. Then $\sum_{j=1}^n a_j + \sum_{j=1}^n b_j = \sum_{j=1}^n c_j$ and for any $1 \leq r_1 < \cdots < r_k \leq n$,*

$\Delta A \quad \Delta A \quad -kb_k$

$$-kb_k + \sum_{j=1}^k b_{n-j+1} \leq \sum_{j=1}^k (c_{r_j} - a_{r_j}) \leq \sum_{j=1}^k b_j - kb_k$$

1) $k=1$, $b_n \leq c_r - a_r \leq b_1$

$b_{n-1} + b_n \leq (c_1+c_3) - (a_1+a_3) \leq b_1 + b_2$

Proof. Suppose $1 \leq r_1 < \cdots < r_k \leq n$. We want to show $\sum_{j=1}^k (c_{r_j} - a_{r_j}) \leq \sum_{j=1}^k b_j$. Replace B by $B - b_k I$. Then each eieganvelue of B and each eigenvalue of $C = A + B$ will be changed by $-b_k$. So, it will not affect the inequalities. Suppose $B = \sum_{j=1}^n b_j \mathbf{x}_j \mathbf{x}_j^*$. Let $B_+ = \sum_{j=1}^k b_j \mathbf{x}_j \mathbf{x}_j^*$. Then

$B \to B - b_k I$

$C = A + B - b_k I$

$$\begin{aligned}
\sum_{j=1}^k (c_{r_j} - a_{r_j}) &\leq \sum_{j=1}^k (\lambda_{r_j}(A + B_+) - \lambda_{r_j}(A)) \quad \text{because} \quad \lambda_j(A+B) \leq \lambda_j(A + B_+) \text{ for all } j \\
&\leq \sum_{j=1}^n (\lambda_j(A + B_+) - \lambda_j(A)) \quad \text{because} \quad \lambda_j(A) \leq \lambda_j(A + B_+) \text{ for all } j \\
&= \operatorname{tr}(A + B_+) - \operatorname{tr}(A) = \sum_{j=1}^k \lambda_j(B_+) = \sum_{j=1}^k b_j.
\end{aligned}$$

A

Replacing (A, B, C) by $(-A, -B, -C)$, we get the other inequalities. $\square$

Wielandt matrix

Lemma 3.3.3 *Suppose $A \in M_{m,n}$ has nonzero singular values $s_1 \geq \cdots \geq s_k$. Then $\begin{pmatrix} 0_m & A \\ A^* & 0_n \end{pmatrix}$ has nonzero eigenvalues $\pm s_1, \dots, \pm s_k$.*

$C = A + B$

Theorem 3.3.4 *Let A, B, C ~~$= A$~~ and $\in M_{m,n}$ with singular values $a_1 \geq \cdots \geq a_n, b_1 \geq \cdots \geq b_n$ and $c_1, \dots, c_n$, respectively. Then*

$$\sum_{j=1}^k (c_{r_j} - a_{r_j}) \leq \sum_{j=1}^k b_j.$$

If $A = U \begin{bmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_k \end{bmatrix} V^*$ ($m \times m$, $n \times n$)

then $\begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}^* \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}$

$\begin{bmatrix} 0 & s_1 \\ s_1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & s_2 \\ s_2 & 0 \end{bmatrix} \oplus \cdots$

Lemma 3.3.3 *Suppose $A \in M_{m,n}$ has nonzero singular values $s_1 \geq \cdots \geq s_k$. Then $\begin{pmatrix} 0_m & A \\ A^* & 0_n \end{pmatrix}$ has nonzero eigenvalues $\pm s_1, \ldots, \pm s_k$.*

with $C = A + B$.

Theorem 3.3.4 *Let $A, B, C \in M_{m,n}$ with singular values $a_1 \geq \cdots \geq a_n, b_1 \geq \cdots \geq b_n$ and $c_1, \ldots, c_n$, respectively. Then*

$$\sum_{j=1}^{k}(c_{r_j} - a_{r_j}) \leq \sum_{j=1}^{k} b_j.$$

Proof:

Then $\begin{bmatrix} 0 & C \\ C^* & 0 \end{bmatrix} = \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} + \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}$.

$c_1 \geq \cdots \geq c_n, 0 \cdots 0$ $a_1 \geq \cdots \geq a_n \geq \cdots 0$ $b_1 \geq \cdots \geq b_n \cdots 0$

$-c_n \geq \cdots \geq -c_n$ $-a_n \cdots -a_1$ $-b_n \geq \cdots \geq -b_1$

3.4 Eigenvalues of principal submatrices

Theorem 3.4.1 *There is a positive semi-definite matrix* $C = \begin{pmatrix} A & * \\ * & B \end{pmatrix}$ *with* $A \in M_k$ *so that* A, B, C *have eigenvalues* $a_1 \geq \cdots \geq a_k$, $b_1 \geq \cdots \geq b_{n-k}$ *and* $c_1 \geq \cdots \geq c_n$, *respectively, if and only if there are positive semi-definite matrices* $\tilde{A}, \tilde{B}, \tilde{C} = \tilde{A} + \tilde{B}$ *with eigenvalues* $a_1 \geq \cdots \geq a_k \geq 0 = a_{k+1} = \cdots = a_n$, $b_1 \geq \cdots \geq b_{n-k} \geq 0 = b_{n-k+1} = \cdots = b_n$, *and* $c_1 \geq \cdots \geq c_n$.

Consequently, ~~for~~ $\sum_{j=1}^{l} (c_{r_j} - a_{r_j}) \leq \sum_{j=1}^{l} b_j$.

$$\sum_{j=1}^{l} c_{r_j} \leq \sum_{j=1}^{l} a_{r_j} + \sum_{j=1}^{l} b_j$$

Remark: In general, if we want to construct

$C = \begin{bmatrix} A & * \\ * & B \end{bmatrix}$. (real symmetric / complex Hermitian)

with prescribed eigenvalues. $c_1 \geq \cdots \geq c_n$

$a_1 \geq \cdots \geq a_k$, $b_1 \geq \cdots \geq b_{n-k}$..

We can change the problem to constructing

$$C + mI = \begin{bmatrix} A+mI & * \\ * & B+mI \end{bmatrix}$$

$~~k~~$ $n-k$

Proof. To prove the necessity, let $\boxed{C = \tilde{C}^*\tilde{C}}$ with $\tilde{C} = [C_1 \; C_2] \in M_n$ with $C_1 \in M_{n,k}$. Then $A = C_1^*C_1$ has eigenvalues $a_1, \dots, a_k$, and $B = C_2^*C_2$ has eigenvalues $b_1, \dots, b_{n-k}$. Now, $\tilde{C} = \tilde{C}\tilde{C}^* = C_1C_1^* + C_2C_2^*$ also eigenvalues $c_1, \dots, c_n$, and $\tilde{A} = C_1C_1^*, \tilde{B} = C_2C_2^*$ have the desired eigenvalues.

Conversely, suppose the $\tilde{A}, \tilde{B}, \tilde{C}$ have the said eigenvalues. Let $\tilde{A} = C_1C_1^*$, $\tilde{B} = C_2C_2^*$ for some $C_1 \in M_{n,k}, C_2 \in M_{n,n-k}$. Then $C = [C_1 \; C_2]^* = [C_1 \; C_2]$ have the desired principal submatrices. □

By the above theorem, one can apply the inequalities governing the eigenvalues of $\tilde{A}, \tilde{B}, \tilde{C} = \tilde{A} + \tilde{B}$ to deduce inequalities relating the eigenvalues of a positive semidefinite matrix C and its complementary principal submatrices. One can also consider general Hermitian matrix by studying $C - \lambda_n(C)I$.

$$\begin{bmatrix} A & * \\ * & B \end{bmatrix} = C = \hat{C}^*\hat{C} = \begin{bmatrix} C_1^* \\ C_2^* \end{bmatrix} \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

$$= \begin{bmatrix} \hat{C}_1^*\hat{C}_1 & * \\ * & \hat{C}_2^*\hat{C}_2 \end{bmatrix}$$

C has e.v. $c_1 \geq \cdots \geq c_n$

$A = \hat{C}_1^*\hat{C}_1$ has e.v. $a_1 \geq \cdots \geq a_k$

$B = \hat{C}_2^*\hat{C}_2$ has e.v. $b_1 \geq \cdots \geq b_{n-k}$

$$\hat{C} = [C_1 | C_2] \begin{bmatrix} C_1^* \\ C_2^* \end{bmatrix} = \underbrace{C_1 C_1^*}_{n \times n} + \underbrace{C_2 C_2^*}_{n \times n}$$

$$= \hat{A} + \hat{B}$$

By the fact that XY and YX have the same non-zero e.v.

$\therefore$ $\hat{C}$ has e.v. $c_1 \geq \cdots \geq c_n$

$\hat{A}$ has e.v. $a_1 \geq \cdots \geq a_k \geq 0 \cdots 0$

$\hat{B}$ has e.v. $b_1 \geq \cdots \geq b_{n-k} \geq 0 \cdots 0$

Suppose $\hat{C} = \hat{A} + \hat{B}$, $\hat{A}$, $\hat{B}$

have eigenvalues:

$$c_1 \geq \cdots \geq c_n, \qquad a_1 \geq \cdots \geq a_k \geq 0 \geq \cdots \geq 0$$

$$b_1 \geq \cdots \geq b_{n-k} \geq 0 \cdots \geq 0.$$

Then
$$\hat{A} = \begin{bmatrix} U \end{bmatrix} \left[\begin{array}{c|c} \begin{matrix} a_1 & & \\ & \ddots & \\ & & a_k \end{matrix} & \\ \hline & 0 \end{array}\right] U^* = {}_{n}\underbrace{\begin{bmatrix} U_1 \end{bmatrix}}_{k} \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_k \end{pmatrix} \begin{bmatrix} U_1^* \end{bmatrix}$$

$$\boxed{= \sum_{i=1}^{k} a_i u_i u_i^*}$$

$$= \underbrace{U_1 \begin{bmatrix} \sqrt{a_1} & & \\ & \ddots & \\ & & \sqrt{a_k} \end{bmatrix}} \underbrace{\begin{bmatrix} \sqrt{a_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{a_k} \end{bmatrix} U_1^*}$$

$$= C_1 \quad C_1^*$$

$$B = V \begin{bmatrix} b_1 & & & \\ & \ddots & & \\ & & b_{n-k} & \\ & & & 0 \end{bmatrix} V^* = \underbrace{V_1}_{n-k} \begin{bmatrix} \sqrt{b_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{b_{n-k}} \end{bmatrix} \begin{bmatrix} & & \end{bmatrix} V_1^*$$

$$= C_2 C_2^*$$

$$\therefore \quad \hat{C} = C_1 C_1^* + C_2 C_2^* = \left[C_1 \,\middle|\, C_2 \right] \begin{bmatrix} C_1^* \\ C_2^* \end{bmatrix}$$

$$\therefore \quad C = \begin{bmatrix} C_1^* \\ C_2^* \end{bmatrix} \begin{bmatrix} C_1 & C_2 \end{bmatrix} = \left[\begin{array}{c|c} C_1^* C_1 & * \\ \hline * & C_2^* C_2 \end{array}\right]$$

Theorem 3.4.2 *There is a Hermitian (real symmetric) matrix $C \in M_n$ with principal submatrix $A \in M_m$ such that C and A have eigenvalues $c_1 \geq \cdots \geq c_n$ and $a_1 \geq \cdots \geq a_m$, respectively, if and only if*

$$\boxed{c_j \geq a_j \quad \text{and} \quad a_{n-j+1} \geq c_{n-j+1}, \qquad j = 1, \ldots, m.}$$

~~C is $n \times n$, A is $(n-1)\times(n-1)$~~

A

~~$a_1 \geq$~~

$$UCU^* = \begin{bmatrix} A & * \\ \hline * & * \end{bmatrix}$$

$$c_1 \geq a_1 \geq c_2 \geq \cdots \geq a_{n-1} \geq c_n$$

$$a_1 \geq \hat{a}_1 \geq \cdots \geq \hat{a}_{n-2} \geq a_{n-1}$$

$c_1 \geq a_1$

$c_2 \geq a_2$

$\vdots$

$c_m \geq a_m$

$a_m \geq c_n$

$a_{m-1} \geq c_{n-1}$

$\vdots$

$a_1 \geq c_{n-m+1}$