Chapter 0 Preliminaries

Objective of the course

Introduce basic matrix results and techniques that are useful in theory and applications.

It is important to know many results, but there is no way I can tell you all of them.

I would like to introduce techniques so that you can obtain the results you want.

This is the standard teaching philosophy of

"It is better to teach students how to fish instead of just giving them fishes."

Assumption You are familiar with

- Linear equations, solution sets, elementary row operations.
- Matrices, column space, row space, null space, ranks,
- Determinant, eigenvalues, eigenvectors, diagonal form.
- Vector spaces, basis, change of bases.
- Linear transformations, range space, kernel.
- Inner product, Gram Schmidt process for real matrices.
- Basic operations of complex numbers.
- Block matrices.

Notation

- $M_n(\mathbb{F})$, $M_{m,n}(\mathbb{F})$ are the set of $n \times n$ and $m \times n$ matrices over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ (or a more general field).
- $\mathbb{F}^n$ is the set of column vectors of length n with entries in $\mathbb{F}$.
- Sometimes we write $M_n, M_{m,n}$ if $\mathbb{F} = \mathbb{C}$.
- If $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \in M_{m+n}(\mathbb{F})$ with $A_1 \in M_m(\mathbb{F})$ and $A_2 \in M_n(\mathbb{F})$, we write $A = A_1 \oplus A_2$.
- x^t, A^t denote the transpose of a vector x and a matrix A.
- For a complex matrix A, $\overline{A}$ denotes the matrix obtained from A by replacing each entry by its complex conjugate. Furthermore, $A^* = (\overline{A})^t$.

More results and examples on block matrix multiplications.

1 Similarity, Jordan form, and applications

1.1 Similarity

Definition 1.1.1 *Two matrices $A, B \in M_n$ are similar if there is an invertible $S \in M_n$ such that $S^{-1}AS = B$, equivalently, $A = T^{-1}BT$ with $T = S^{-1}$. A matrix A is diagonalizable if A is similar to a diagonal matrix.*

Remark 1.1.2 *If A and B are similar, then $\det(A) = \det(B)$ and $\det(zI - A) = \det(zI - B)$. Furthermore, if $S^{-1}AS = B$, then $Bx = \mu x$ if and only if $A(Sx) = \mu(Sx)$.*

Theorem 1.1.3 *Let $A \in M_n(\mathbb{F})$. Then A is diagonalizable over $\mathbb{F}$ if and only if it has n linearly independent eigenvectors in $\mathbb{F}^n$.*

Proof. An invertible matrix $S \in M_n$ satisfies $S^{-1}AS = D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ if and only if $AS = SD$, i.e., the columns of S equal to $x_1, \ldots, x_n$. □

Remark 1.1.4 *Suppose $A = SDS^{-1}$ with $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. Let S have columns $x_1, \ldots, x_n$ and S^{-1} have rows $y_1^t, \ldots, y_j^t$. Then*

$$Ax_j = \lambda_j x_j \quad \text{and} \quad y_j^t A = \lambda_j y_j^t, \qquad j = 1, \ldots, n,$$

and

$$A = \sum_{j=1}^{n} \lambda_j x_j y_j^t.$$

Thus, $x_1, \ldots, x_n$ are the (right) eigenvectors of A, and $y_1^t, \ldots, y_n^t$ are the left eigenvectors of A.

Remark 1.1.5 *For a real matrix $A \in M_n(\mathbb{R})$, we may not be able to find real eigenvalues, and we may not be able to find n linearly independent eigenvectors even if $\det(zI - A) = 0$ has only real roots.*

Example 1.1.6 *Consider $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then A has no real eigenvalues, and B does not have two linearly independent eigenvectors.*

Remark 1.1.7 *By the fundamental theorem of algebra, every complex polynomial can be written as a product of linear factors. Consequently, for every $A \in M_n(\mathbb{C})$, the characteristic polynomial has linear factors:*

$$\det(zI - A) = (z - \lambda_1) \cdots (z - \lambda_n).$$

So, we needs only to find n linearly independent eigenvectors in order to show that A is diagonalizable.

1.2 Triangular form and Jordan form

Question What simple form can we get by similarity if $A \in M_n$ is not diagonalizable?

Theorem 1.2.1 *Let $A \in M_n$ be such that $\det(zI - A) = (z-\lambda_1)\cdots(z-\lambda_n)$. For any permutation $(j_1, \ldots, j_n)$ of $(1, \ldots, n)$, A is similar to a matrix in (upper) triangular form with diagonal entries $\lambda_{j_1}, \ldots, \lambda_{j_n}$.*

Proof. By induction. The result clearly holds if $n = 1$. Suppose $Ax_1 = \lambda_{i_1} x_1$. Let S_2 be invertible with first column equal to x_1. Then $AS_1 = S_1 \begin{pmatrix} \lambda_{i_1} & * \\ 0 & A_1 \end{pmatrix}$. Thus, $S_1^{-1} A S_1 = \begin{pmatrix} \lambda_{i_1} & * \\ 0 & A_1 \end{pmatrix}$. Note that the eigenvalues of A_1 are $\lambda_{i_2}, \ldots, \lambda_{i_n}$. By induction assumption, $S_2^{-1} A_1 S_2$ is in upper triangular form with eigenvalues $\lambda_{i_2}, \ldots, \lambda_{i_n}$. Let $S = S_1([1] \oplus S_2)$. Then $S^{-1}AS$ has the required form. □

Note that we can apply the argument to A^t to get an invertible T such that $T^{-1}A^tT$ is in upper triangular form. So, $S^{-1}AS$ is in lower triangular form for $S = T^{-1}$.

Lemma 1.2.2 *Suppose $A \in M_m, B \in M_n$ have no common eigenvalues. Then for any $C \in M_{m,n}$ there is $X \in M_{m,n}$ such that $AX - XB = C$.*

Proof. Suppose $S^{-1}AS = \tilde{A}$ is in upper triangular form, and $T^{-1}BT = \tilde{B}$ is in lower triangular form. Multiplying the equation $AX - XB = C$ on left and right by S and T^{-1}, we get $\tilde{A}Y - Y\tilde{B} = \tilde{C}$ with $Y = SXT^{-1}$ and $\tilde{C} = SYT^{-1}$. Once solve $\tilde{A}Y - Y\tilde{B} = \tilde{C}$, we get the solution $X = S^{-1}YT$ for the original problem.

Now, for each column $\mathrm{col}_j(\tilde{C})$ for $j = 1, \ldots, n$, we have $\mathrm{col}_j(Y) - \sum_{i=1}^n \tilde{B}_{ij}\mathrm{col}_i(Y) = \mathrm{col}_j(\tilde{C})$. We get a system of linear equation with coefficient matrix

$$I_n \otimes \tilde{A} - \tilde{B} \otimes I_m = \begin{pmatrix} A & & \\ & \ddots & \\ & & A \end{pmatrix} - \begin{pmatrix} \tilde{B}_{11} I_m & \ldots & b_{n1} I_m \\ \vdots & \ddots & \vdots \\ \tilde{B}_{n1} I_m & \ddots & \tilde{B}_{nn} I_m \end{pmatrix}.$$

Because $\tilde{B}$ is in lower triangular form, and the upper triangular matrix $\tilde{A} = S^{-1}AS$ and the lower triangular matrix $\tilde{B} = T^{-1}BT$ have no common eigenvalues, the coefficient matrix is in upper triangular form with nonzero diagonal entries. So, the equation $\tilde{A}Y - Y\tilde{B} = \tilde{C}$ always has a unique solution. □

Theorem 1.2.3 *Suppose $A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \in M_n$ such that $A_{11} \in M_k, A_{22} \in M_{n-k}$ have no common eigenvalue. Then A is similar to $A_{11} \oplus A_{22}$.*

Proof. By the previous lemma, there is X be such that $A_{11}X + A_{12} = XA_{22}$. Let $S = \begin{pmatrix} I_k & X \\ 0 & I_{n-k} \end{pmatrix}$ so that $AS = S(A_{11} \oplus A_{22})$. The result follows. □

Corollary 1.2.4 *Suppose $A \in M_n$ has distinct eigenvalues $\lambda_1, \dots, \lambda_k$. Then A is similar to $A_{11} \oplus \cdots \oplus A_{kk}$ such that A_{jj} has (only one distinct) eigenvalue λ_j for $j = 1, \dots, k$.*

Definition 1.2.5 *Let $J_k(\lambda) \in M_k$ such that all the diagonal entries equal λ and all super diagonal entries equal 1. Then $J_k(\lambda) = \begin{pmatrix} \lambda & 1 & & & \\ & \ddots & \ddots & & \\ & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix} \in M_k$ is call a (an upper triangular) Jordan block of λ of size k.*

Theorem 1.2.6 *Every $A \in M_n$ is similar to a direct sum of Jordan blocks.*

Proof. We may assume that $A = A_{11} \oplus \cdots \oplus A_{kk}$. If we can find invertible matrices $S_1, \dots, S_k$ such that $S_i^{-1} A_{ii} S_i$ is in Jordan form, then $S^{-1}AS$ is in Jordan form for $S = S_1 \oplus \cdots \oplus S_k$.

Focus on $T = A_{ii} - \lambda_i I_{n_k}$. If $S^{-1}TS$ is in Jordan form, then so is A_{ii}. Now, we use a proof of Mark Wildon.

https://www.math.vt.edu/people/renardym/class_home/Jordan.pdf

Note: Then vectors $u_1, Tu_1, \dots, T^{n_1}u_1$ in the proof is called a Jordan chain. □

Example 1.2.7 *Let $T = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then $Te_1 = 0, Te_2 = 0, Te_3 = e_1 + 3e_2, Te_4 = 2e_1 + 4e_2$. So, $T(V) = \operatorname{span}\{e_1, e_2\}$. Now, $Te_1 = Te_2 = 0$ so that e_1, e_2 form a Jordan basis for $T(V)$. Solving u_1, u_2 such that $T(u_1) = e_1, T(u_2) = e_2$, we let $u_1 = -2e_3 + 3e_4/2$ and $u_2 = e_3 - e_4/2$. Thus, $TS = S(J_2(0) \oplus J_2(0))$ with*

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \\ 0 & 3/2 & 0 & -1/2 \end{pmatrix}.$$

Example 1.2.8 *Let $T = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$. Then $Te_1 = 0, Te_2 = e_1, Te_3 = 2e_1 + e_2$. So, $T(V) = \operatorname{span}\{e_1, e_2\}$, and $e_2, Te_2 = e_1$ form a Jordan basis for $T(V)$. Solving u_1 such that $T(u_1) = e_2$, we* *have* $u_1 = (-2e_2 + e_3)/3$. *Thus,* $TS = SJ_3(0)$ *with*

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1/3 \end{pmatrix}.$$

Remark 1.2.9 *The Jordan form of* A *can be determined by the rank/nullity of* $(A - \lambda_j I)^k$ *for* $k = 1, 2, \ldots$. *So, the Jordan blocks are uniquely determined.*

Let $\ker((A - \lambda I)^i) = \ell_i$ has dimension ℓ_i. Then there are ℓ_1 Jordan blocks of λ, and there are $\ell_i - \lambda_{i-1}$ Jordan blocks of size at least i.

Example 1.2.10 *Suppose* $A \in M_9$ *has distinct eigenvalues* $\lambda_1, \lambda_2, \lambda_3$ *such that* $A - \lambda_1 I$ *has rank 8,* $A - \lambda_2 I$ *has rank 7,* $(A - \lambda_2 I)^2$ *and* $(A - \lambda_2 I)^3$ *have rank 5,* $A - \lambda_3 I$ *has rank 6,* $(A - \lambda_3 I)^2$ *and* $(A - \lambda_3 I)^3$ *have rank 5. Then the Jordan form of* A *is*

$$J_1(\lambda_1) \oplus J_2(\lambda_2) \oplus J_2(\lambda_2) \oplus J_1(\lambda_3) \oplus J_1(\lambda_3) \oplus J_2(\lambda_3).$$

1.3 Implications of the Jordan form

Theorem 1.3.1 *Two matrices are similar if and only if they have the same Jordan form.*

Proof. If A and B have Jordan form J, then $S^{-1}AS = J = T^{-1}BT$ for some invertible S, T so that $R^{-1}AR = B$ with $R = ST^{-1}$.

If $S^{-1}AS = B$, then $\operatorname{rank}(A - \mu I)^\ell = \operatorname{rank}(B - \mu I)^\ell$ for all eigenvalues of A or B, and for all positive integers ℓ. So, A and B have the same Jordan form. $\square$

Remark 1.3.2 *If $A = S(J_1 \oplus \cdots \oplus J_k)S^{-1}$, then $A^m = S(J_1^m \oplus \cdots \oplus J_k^m)S^{-1}$.*

Theorem 1.3.3 *Let $J_k(\lambda) = \lambda I_k + N_k$, where $N_k = \sum_{j=1}^{k-1} E_{j,j+1}$. Then $J_k(\lambda)^m = \sum_{j=0}^{m} \binom{m}{j} \lambda^{m-j} N_k^j$, where $N_k^0 = I_k$, $N_k^j = 0$ for $j \geq k$, and N_k^j has one's at the jth super diagonal (entries with indexes $(\ell, \ell + j)$) and zeros elsewhere.*

For every polynomial function $f(z) = a_m z^m + \cdots + a_0$, let

$$f(A) = a_m A^m + \cdots + a_0 I_n \quad \text{for } A \in M_n.$$

Theorem 1.3.4 (Cayley-Hamilton) *Let $A \in M_n$ and $f(z) = \det(zI - A)$. Then $f(A) = 0$.*

Proof. Let $f(z) = \det(zI - A) = (z - \lambda_1)^{n_1} \cdots (z - \lambda_k)^{n_k}$ such that $n_1 + \cdots + n_k$. Then there is an invertible S such that $S^{-1}AS = A_{11} \oplus \cdots \oplus A_{kk}$ with $A_{jj} \in M_{n_k}$ is a direct sum of Jordan blocks of λ_j Suppose the maximum size of the Jordan block in A_{jj} is r_j, then $r_j \leq n_j$ for $j = 1, \ldots, k$. Thus,

$$f(A_{jj}) = (A_{jj} - \lambda_1 I)^{n_1} \ldots (A - \lambda_k)^{n_k} = 0.$$

As a result,

$$f(A) = f(S(A_{11} \oplus \cdots \oplus A_{kk})S^{-1}) = S[f((A_{11}) \oplus \cdots \oplus f(A_{kk})]S^{-1} = 0_n.$$

$\square$

Definition 1.3.5 *Let $A \in M_n$. Then there is a unique monic polynomial*

$$m_A(z) = x^m + a_1 x^{m-1} + \cdots + a_m$$

such that $m_A(A) = 0$. It is the minimal polynomial of A.

Theorem 1.3.6 *A polynomial $g(z)$ satisfies $g(A) = 0$ if and only if it is a multiple of the minimal polynomial of A.*

Proof. If $g(z) = m_A(z)q(z)$, then $g(A) = m_A(A)q(A) = 0$. To prove the converse, by the Euclidean algorithm, $g(z) = m_A(z)q(z)+r(z)$ for any polynomial $g(z)$. If $0 = g(A) = m_A(A)q(A)+r(A) = r(A)$, then $r(A) = 0$. But $r(z)$ has degree less than $m_A(z)$. If $r(z)$ is not zero, then there is a nonzero $\mu \in \mathbb{C}$ such that $\mu r(z)$ is a monic polynomial such that $\mu r(A) = 0$, which is impossible. So, $r(z) = 0$, i.e., $g(z)$ is a multiple of $m_A(z)$. □

We can actually determine the minimal polynomial of $A \in M_n$ using its Jordan form.

Theorem 1.3.7 *Suppose A has distinct eigenvalues $\lambda_1, \dots, \lambda_k$ such that r_j is the maximum size Jordan block of λ_j for $j = 1, \dots, k$. Then $m_A(z) = (z-\lambda_1)^{r_1} \cdots (z-\lambda_k)^{m_k}$.*

Proof. Following the proof of the Cayley Hamilton Theorem, we see that $m_A(A) = 0_n$. By the last Theorem, if $g(A) = 0_n$, then $g(z) = m_A(z)q(z)$. So, taking $q(z) = 1$ will yield the monic polynomial of minimum degree satisfying $m_A(A) = 0$. □

Remark 1.3.8 *For any polynomial $g(z)$, the Jordan form of $g(A)$ can be determine in terms of the Jordan form of A. In particular, for every Jordan block $J_k(\lambda)$, we can write $g(z) = (z-\lambda)^k q(z)+r(z)$ with $r(z) = a_0 + \cdots + a_{k-1}z^{k-1}$ so that $g(J_k(\lambda)) = r(J_k(\lambda))$.*

1.4 Final remarks

- If $A \in M_n(\mathbb{R})$ has only real eigenvalues, then one can find a real invertible matrix such that $S^{-1}AS$ is in Jordan form.
- If $A \in M_n(\mathbb{R})$, then there is a real invertible matrix such that $S^{-1}AS$ is a direct sum of real Jordan blocks, and $2k \times 2k$ generalized Jordan blocks of the form $(C_{ij})_{1\le i,j\le k}$ with $C_{11} = \cdots = C_{kk} = \begin{pmatrix} \mu_1 & \mu_2 \\ -\mu_2 & \mu_1 \end{pmatrix}$, $C_{12} = \cdots = C_{k-1,k} = I_2$, and all other blocks equal to 0_2.
- The proof can be done by the following two steps.

  First of all find the Jordan form of A. Then group $J_k(\lambda)$ and $J_k(\bar{\lambda})$ together for any complex eigenvalues, and find a complex S such that $S^{-1}AS$ is a direct sum of the above form.

  Second if $S = S_1 + iS_2$ for some real matrix S_1, S_2, show that there is $\hat{S} = S_1 + rS_2$ for some real number r such that $\hat{S}$ is invertible so that $\hat{S}^{-1}A\hat{S}$ has the desired form.

Another proof of the Jordan canonical form theorem is to use λ-matrices. Let $A \in M_n$. Then one can apply elementary row and column reductions using elementary matrices in

$$\{(f_{ij}(\lambda)) : f_{ij}(\lambda) \text{ is a plynomial in } \lambda\}$$

to get the Smith normal form

$$\operatorname{diag}(d_1(\lambda), \dots, d_k(\lambda), 0, \dots, 0)$$

for some monic polynomials $d_j(\lambda)$ with $d_j(\lambda)|d_{j+1}(\lambda)$ for $j = 1, \ldots, k-1$.

Furthermore, the following are true.

- Two matrices $A, B \in M_n$ are similar if and only if there are λ-matrices $P(\lambda), Q(\lambda)$ such that $P(\lambda)(\lambda I - A) = (\lambda I - B)Q(\lambda)$.
- Setting $\lambda = 0$, one has $P_0 A = BQ_0$ and that $P_0 Q_0 = I$.
- For every $A \in M_n$, there is B is Jordan form so that $P_0 A = BQ_0$.

2 Unitary similarity, unitary equivalence, and consequences

2.1 Unitary space and unitary matrices

Definition 2.1.1 *Let* $\mathbf{x} = (x_1, \ldots, x_n)^t, \mathbf{y} = (y_1, \ldots, y_n)^t$. *Define the inner product of* $\mathbf{x}$ *and* $\mathbf{y}$ *by*

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{j=1}^{n} x_j \bar{y}_j.$$

The vectors are orthogonal if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$. *A set* $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ *is orthonormal if* $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \delta_{ij}$. *A matrix* $U \in M_n$ *with orthonormal columns is a unitary matrix, i.e.,* $U^*U = I_n$. *A basis for* $\mathbb{C}^n$ *is an orthonormal basis if it consists of orthonormal vectors.*

Some basic properties For any $a, b \in \mathbb{C}, \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{C}^n$,

- $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$.
- $\langle a\mathbf{x} + b\mathbf{y}, \mathbf{z} \rangle = a\langle \mathbf{x}, \mathbf{y} \rangle + b\langle \mathbf{x}, \mathbf{z} \rangle$,
- $\langle \mathbf{x}, a\mathbf{y} + b\mathbf{z} \rangle = \bar{a}\langle \mathbf{x}, \mathbf{y} \rangle + \bar{b}\langle \mathbf{x}, \mathbf{z} \rangle$,
- $\langle \mathbf{x}, \mathbf{x} \rangle = \sum_{j=1} |x_j|^2$ if $\mathbf{x} = (x_1, \ldots, x_n)^t$.
- $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, where the equality holds if and only if $\mathbf{x} = \mathbf{0}$.

Theorem 2.1.2 (a) *Orthonormal sets in* $\mathbb{C}^n$ *are linearly independent. If* $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ *is an orthonormal basis for* $\mathbb{C}^n$, *then for any* $\mathbf{v} \in \mathbb{C}^n$, $\mathbf{v} = c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n$ *with* $c_j = \langle \mathbf{v}, \mathbf{u}_j \rangle$ *for* $j = 1, \ldots, n$.

(b) [Gram Schmidt process] *Suppose* $A = M_{n,m}$ *has linearly independent columns. Then* $A = VR$ *such that* $V \in M_{n,m}$ *has orthonormal column, and* $R \in M_m$ *is upper triangular.*

(c) *Every linearly independent (orthonormal) subset of* $\mathbb{C}^n$ *can be extended to a (an orthonormal) basis.*

Proof. (a) Suppose $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthonormal set, and $c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k = 0$. Then

$$0 = \langle \sum c_j \mathbf{u}_j, \mathbf{u}_i \rangle = c_i \qquad \text{for all } i = 1, \ldots, k.$$

Let $k = n$, and $\mathbf{v} = c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n$. Then $c_i = \langle \mathbf{v}, \mathbf{u}_i \rangle$ for all $i = 1, \ldots, n$.

(b) Suppose A has column $\mathbf{a}_1, \ldots, \mathbf{a}_m$. Let

$$\mathbf{u}_1 = \mathbf{a}_1, \;\; \mathbf{u}_2 = \mathbf{a}_2 - \frac{\langle \mathbf{a}_2, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle}\mathbf{u}_1, \;\; \mathbf{u}_3 = \mathbf{a}_3 - \frac{\langle \mathbf{a}_3, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle}\mathbf{u}_1 - \frac{\langle \mathbf{a}_3, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle}\mathbf{u}_2, \;\; \ldots.$$

Then set $\mathbf{v}_j = \mathbf{u}_j/\|\mathbf{u}_j\|$ for $j = 1, \ldots, m$, and V be $n \times m$ with columns $\mathbf{v}_1, \ldots, \mathbf{v}_m$. We have $A = VR$, where $R = V^*A$ is upper triangular.

(c) Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be linearly independent. Take the pivoting columns of $[\mathbf{v}_1 \cdots \mathbf{v}_k \; \mathbf{e}_1 \cdots \mathbf{e}_n]$ to form an invertible matrix, and then apply Gram Schmidt process.

2.2 Schur Triangularization

Theorem 2.2.1 *For every matrix $A \in M_n$ with eigenvalues $\mu_1, \ldots, \mu_n$ (listed with multipicities),there is a unitary U such that U^*AU is in upper (or lower) triangular form with diagonal entries $\mu_1, \ldots, \mu_n$.*

Proof. Similar to that of Theorem 1.2.1.

Definition 2.2.2 *Let $A \in M_n$. The matrix A is normal if $AA^* = A^*A$, the matrix A is Hermitian if $A = A^*$, the matrix A is positive semidefinite if $\mathbf{x}^*A\mathbf{x} \geq 0$ for all $x \in \mathbb{C}^n$, the matrix A the matrix A is positive definite if $\mathbf{x}^*A\mathbf{x} > 0$ for all nonzero $x \in \mathbb{C}^n$.*

Theorem 2.2.3 *Let $A \in M_n$.*

(a) *The matrix A is normal if and only if it is unitarily diagonalizable.*

(b) *The matrix A is unitary if and only if it is unitarily similar to a diagonal matrix with all eigenvalues having modulus 1.*

Proof. (a) If $U^*AU = D$, i.e., $A = UDU^*$ for some unitary $U \in M_n$. Then $AA^* = UDU^*UD^*U = UDD^*U^* = UD^*DU^* = UD^*U^*UDU^* = A^*A$.

Conversely, suppose $U^*AU = (a_{ij}) = \tilde{A}$ is in upper triangular form. If $AA^* = A^*A$, then $\tilde{A}\tilde{A}^* = \tilde{A}^*\tilde{A}$ so that the $(1,1)$ entries of the matrices on both sides are the same. Thus,

$$|a_{11}|^2 + \cdots + |a_{1n}|^2 = |a_{11}|^2$$

implying that $\tilde{A} = [a_{11}] \oplus A_1$, where $A_1 \in M_{n-1}$ is in upper triangular form. Now,

$$[|a_{11}|^2] \oplus A_1A_1^* = \tilde{A}\tilde{A}^* = \tilde{A}^*\tilde{A} = [|a_{11}|^2] \oplus A_1^*A_1.$$

Consider the $(1,1)$ entries of $A_1A_1^*$ and $A_1^*A_1$, we see that all the off-diagonal entries in the second row of A_1 are zero. Repeating this process, we see that $\tilde{A} = \operatorname{diag}(a_{11}, \ldots, a_{nn})$.

(b) If $U^*AU = D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ with $|\lambda_1| = \cdots = |\lambda_n| = 1$, then A is unitary because

$$AA^* = UDU^*UD^*U^* = U(DD^*)U^* = UU^* = I_n.$$

Conversely, if $AA^* = A^*A = I_n$, then $U^*AU = D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ for some unitary $U \in M_n$. Thus, $I = U^*IU = U^*AUU^*A^*U = DD^*$. Thus, $|\lambda_1| = \cdots = |\lambda_n| = 1$. □

Theorem 2.2.4 *Let $A \in M_n$. The following are equivalent.*

(a) *A is Hermitian.*

(b) *A is unitarily similar to a real diagonal matrix.*

(c) $\mathbf{x}^* A\mathbf{x} \in \mathbb{R}$ *for all* $\mathbf{x} \in \mathbb{C}^n$.

Proof. Suppose (a) holds. Then $AA^* = A^2 = A^*A$ so that $U^*AU = D = \text{diag}\,(\lambda_1, \dots, \lambda_n)$ for some unitary $U \in M_n$. Now, $D = U^*AU = U^*A^*U = (U^*AU)^* = D^*$. So, $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. Thus (b) holds.

Suppose (b) holds and $A = U^*DU$ such that U is unitary and $D = \text{diag}\,(d_1, \dots, d_n$. Then for any $\mathbf{x} \in \mathbb{C}^n$, we can set $U\mathbf{x} = (y_1, \dots, y_n)^t$ so that $\mathbf{x}^*A\mathbf{x} = \mathbf{x}^*U^*DU\mathbf{x} = \sum_{j=1}^n d_j|y_j|^2 \in \mathbb{R}$.

Suppose (c) holds. Let $A = H + iG$ with $H = (A + A^*)/2 \geq 0$ and $G = (A - A^*)/(2i)$. Then $H = H^*$ and $G = G^*$. Then for any $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{x}^*H\mathbf{x} = \mu_1 \in \mathbb{R}, \mathbf{x}^*G\mathbf{x} = \mu_2 \in \mathbb{R}$ so that $\mathbf{x}^*A\mathbf{x} = \mu_1 + i\mu_2 \in \mathbb{C}$. If G is nonzero, then $V^*GV = \text{diag}\,(\lambda_1, \dots, \lambda_n)$ with $\lambda_1 \neq 0$. Suppose $\mathbf{x}$ is the first column of V. Then $\mathbf{x}^*A\mathbf{x} = \mathbf{x}^*H\mathbf{x} + i\mathbf{x}^*G\mathbf{x} = \mu_1 + i\lambda_1 \notin \mathbb{R}$, which is a contradiction. So, we have $G = 0$ and $A = H$ is Hermitian. □

Theorem 2.2.5 *Let* $A \in M_n$. *The following are equivalent.*

(a) A *is positive semidefinite.*

(b) A *is unitarily similar to a real diagonal matrix with nonnegative diagonal entries.*

(c) $A = B^*B$ *for some* $B \in M_n$. *(We can choose* B *so that* $B = B^*$.*)*

Proof. Suppose (a) holds. Then $\mathbf{x}^*A\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{C}^n$. Thus, there is a unitary $U \in M_n$ such that $U^*AU = \text{diag}\,(\lambda_1, \dots, \lambda_n)$ with $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. If there is $\lambda_j < 0$, we can let $\mathbf{x}$ be the jth column of U so that $\mathbf{x}^*A\mathbf{x} = \lambda_j < 0$, which is a contradiction. So, all $\lambda_1, \dots, \lambda_n \geq 0$.

Suppose (b) holds. Then $U^*AU = D$ such that D has nonnegative entries. We have $A = B^*B$ with $B = UD^{1/2}U^* = B^*$. Hence condition (c) holds.

Suppose (c) holds. Then for any $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{x}^*A\mathbf{x} = (B\mathbf{x})^*(B\mathbf{x}) \geq 0$. Thus, (a) holds. □

For any $A \in M_n$ we can write $A = H + iG$ with $H = (A + A^*)/2$ and $G = (A - A^*)/(2i)$. This is known as the Hermitian or Cartesian decomposition.

2.3 Spetch's theorem and Commuting families

There is no easy canonical form under unitary similarity. [1] How to determine two matrices are unitarily similar?

Definition 2.3.1 *Let* $\{X, Y\} \subseteq M_n$. *A word* $W(X, Y)$ *in* X *and* Y *of length* m *is a product of* m *matrices chosen from* $\{X, Y\}$ *(with repetition).*

[1] Helene Shapiro, A survey of canonical forms and invariants for unitary similarity, Linear Algebra Appl. 147 (1991), 101-167.

Theorem 2.3.2 *Let $A, B \in M_n$.*

(a) *If A and B are unitarily similar, then $\mathrm{tr}\,(W(A, A^*)) = \mathrm{tr}\,(W(B, B^*))$ for all words $W(X, Y)$.*

(b) $\mathrm{tr}\,(W(A, A^*)) = \mathrm{tr}\,(W(B, B^*))$ *for all words $W(X, Y)$ of length $2n^2$, then A and B are unitarily similar.*

Definition 2.3.3 *A family $\mathcal{F} \subseteq M_n$ is a commuting family if every pair of matrices $X, Y \in \mathcal{F}$ commute, i.e., $XY = YX$.*

Lemma 2.3.4 *Let $\mathcal{F} \subseteq M_n$ be a commuting family. Then there a unit vector $v \in \mathbb{C}^n$ such that v is an eigenvector for every $A \in \mathcal{F}$.*

Proof. Let $V \subseteq \mathbb{C}^n$ with minimum dimension be such that $A(V) \subseteq V$. We will show that $\dim V = 1$ and the result will follow. Clearly, $\dim V \leq n$. Suppose $\dim V > 1$. We claim that every nonzero vector in V is an eigenvector of A for every $A \in \mathcal{F}$. If it is not true, let $A \in \mathcal{F}$ so that not every nonzero vector in V is an eigenvector of A. Since $A(V) \subseteq V$, there is $v \in V$ such that $Av = \lambda v$. Let $V_0 = \{u \in V : Au = \lambda u\} \subset V$. Note that for any $B \in \mathcal{F}$ and $u \in V_0$, we have Bu satisfying $A(Bu) = BAu = B\lambda u = \lambda Bu$, i.e., $Bu \in V_0$. So, V_0 satisfies $B(V_0) \subseteq V_0$ and $\dim V_0 < \dim V$, which is impossible. The desired result follows. □

Theorem 2.3.5 *Let $\mathcal{F} \subseteq M_n$. Then there is an invertible matrix $S \in M_n$ such that $S^{-1}AS$ is in upper triangular form. The matrix S can be chosen to be unitary.*

Proof. Similar to the proof of Theorem 1.2.1. □

Corollary 2.3.6 *Suppose $\mathcal{F} \subseteq M_n$ is a commuting family of normal matrices. Then there is a unitary matrix $U \in M_n$ such that U^*AU is in diagonal form.*

2.4 Singular decomposition and polar decomposition

Lemma 2.4.1 *Let A be a nonzero $m \times n$ matrix, and $u \in \mathbb{C}^m, v \in \mathbb{C}^n$ be unit vectors such that $|u^*Av|$ attains the maximum value. Suppose $U \in M_m$ and $V \in M_n$ are unitary matrices with u and v as the first columns, respectively. Then $U^*AV = \begin{pmatrix} u^*Av & 0 \\ 0 & A_1 \end{pmatrix}$.*

Proof. Note that the existence of the maximum $|u^*Av|$ follows from basis analysis result.

Suppose $U^*AV = (a_{ij})$. If the first column $x = (a_{11}, \ldots, a_{m1})^t$ has nonzero entries other than a_{11}, then $\tilde{u} = Ux/\|x\| \in \mathbb{C}^m$ is a unit vector such that

$$\tilde{u}^*Av = x^*UAv/\|x\| = x^*x/\|x\| > |a_{11}|^2,$$

which contradicts the choice of u and v. Similarly, if the first row $y^* = (a_{11}, \ldots, a_{1n})$ has nonzero entries other than a_{11}, then $\tilde{v} = Vy/\|y\| \in \mathbb{C}^n$ is a unit vector satisfying

$$u^* A\tilde{v} = u^* AVy/\|y\| = y^* y/\|y\| > |a_{11}|^2,$$

which is a contradiction. The result follows. □

Theorem 2.4.2 *Let A be an $m \times n$ matrix. Then there are unitary matrices $U \in M_m, V \in M_n$ such that*

$$U^* AV = D = \sum_{j=1}^{k} s_j E_{jj}.$$

Proof. We prove the result by induction on $\max\{m, n\}$. By the previous lemma, there are unitary matrices $U \in M_m, V \in M_n$ such that $U^* AV = \begin{pmatrix} u^* Av & 0 \\ 0 & A_1 \end{pmatrix}$. We may replace U by $e^{ir} U$ for a suitable $r \in [0, 2\pi)$ and assume that $u^* Av = |u^* Av| = s_1$. By induction assumption there are unitary matrices $U_1 \in M_{m-1}, V_1 \in M_{n-1}$ such that $U_1^* A_1 V_1 = \begin{pmatrix} s_2 & & \\ & s_3 & \\ & & \ddots \end{pmatrix}$. Then $([1] \oplus U_1^*) U^* AV ([1] \oplus V_1)$ has the asserted form. □

Remark 2.4.3 *Let U_1 be formed by the first k columns $\mathbf{u}_1, \ldots, \mathbf{u}_k$ of U and V_1 be formed by the first k columns $\mathbf{v}_1, \ldots, \mathbf{v}_k$ of V. Then*

$$A = U_1 \text{diag}\,(s_1, \ldots, s_k) V_1^* = \sum_{j=1}^{k} s_j \mathbf{u}_j \mathbf{v}_j^*.$$

The values $s_1 \geq \cdots \geq s_k > 0$ are the nonzero **singular values** of A, which are $s_1^2, \ldots, s_k^2$ are the nonzero eigenvalues of AA^* and $A^* A$. The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are the right singular vectors of A, and $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are the left singular vectors of A. So, they are uniquely determined.

Here is another way to determine the singular value decomposition. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subseteq \mathbb{C}^n$ be an orthonormal set of eigenvectors corresponding to the nonzero eigenvalues $s_1^2, \ldots, s_k^2$ of $A^* A$. Let $\mathbf{u}_j = A\mathbf{v}_j / s_j$. Then $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\} \subseteq \mathbb{C}^m$ is an orthonormal family such that $A = \sum_{j=1}^k s_j \mathbf{u}_j \mathbf{v}_j^*$.

Similarly, let $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\} \subseteq \mathbb{C}^m$ be an orthonormal set of eigenvectors corresponding to the nonzero eigenvalues $s_1^2, \ldots, s_k^2$ of AA^*. Let $v_j = A^* \mathbf{u}_j / s_j$. Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subseteq \mathbb{C}^n$ is an orthonormal family such that $A = \sum_{j=1}^k s_j \mathbf{u}_j \mathbf{v}_j^*$.

Corollary 2.4.4 *Let $A \in M_n$. Then $A = U^* P = QV$ such that $U, V \in M_n$ are unitary, and P, Q are positive semidefinite matrices with eigenvalues equal to the singular values of A.*

2.5 Other canonical forms

We have considered the canonical forms under similarity and unitary similarity. Here, we consider other canonical forms for different classes of matrices.

Unitary equivalence

- Two matrices $A, B \in M_{m,n}$ are unitarily equivalent if there are unitary matrices $U \in M_m, V \in M_n$ such that $U^*AV = B$.
- Every matrix $A \in M_{m,n}$ is equivalent to $\sum_{j=1}^k s_j E_{jj}$, where $s_1 \geq \cdots \geq s_k > 0$ are the nonzero singular values of A.
- Two matrices are unitarily equivalent if and only if they have the same singular values.

Equivalence

- Two matrices $A, B \in M_{m,n}$ are equivalent if there are invertible matrices $R \in M_m, S \in M_n$ such that $A = RBS$.
- Every matrix $A \in M_{m,n}$ is equivalent to $\sum_{j=1}^k E_{jj}$, where k is the rank of A.
- Two matrices are equivalent if they have the same rank.

Proof. Elementary row operations and elementary column operations. □

$*$-congruence

- A matrix $A \in M_n$ is $*$-congruent to $B \in M_n$ if there is an invertible matrix S such that $A = S^*BS$.
- There is no easy canonical form under $*$-congruence for general matrix.[2]
- Every Hermitian matrix $A \in M_n$ is $*$-congruent to $I_p \oplus -I_q \oplus 0_{n-p-q}$. The triple $\nu(A) = (p, q, n-p-q)$ is known as the inertia of A.
- Two Hermitian matrices are $*$-congruent if and only if they have the same inertia.

Proof. Use the unitary congruence/similarity results. □

Congruence or t-congruence

- A matrix $A \in M_n$ is t-congruent to $B \in M_n$ if there is an invertible matrix S such that $A = S^tBS$.

[2] Roger A. Horn and Vladimir V. Sergeichuk, Canonical forms for complex matrix congruence and $*$-congruence, Linear Algebra Appl. (2006), 1010-1032.

- There is no easy canonical form under t-congruence for general matrices; see footnote 2.
- Every complex symmetric matrix $A \in M_n$ is t-congruent to $I_k \oplus 0_{n-k}$, where $k = \text{rank}\,(A)$.
- Every skew-symmetric $A \in M_n$ is t-congruent to 0_{n-2k} and k copies of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
- The rank of a skew-symmetric matrix $A \in M_n$ is even.
- Two symmetric (skew-symmetric) matrices are t-congruent if and only if they have the same rank.

Proof. Use the unitary congruence results. □

Unitary congruence

- A matrix $A \in M_n$ is unitarily congruent to $B \in M_n$ if there is a unitary matrix U such that $A = U^t BU$.
- There is no easy canonical form under unitary congruence for general matrices.
- Every complex symmetric matrix $A \in M_n$ is unitarily congruent to $\sum_{j=1}^{k} s_j E_{jj}$, where $s_1 \geq \cdots \geq s_k > 0$ are the nonzero singular values of A.
- Every skew-symmetric $A \in M_n$ is unitarily congruent to 0_{n-2k} and

  $$\begin{pmatrix} 0 & s_j \\ -s_j & 0 \end{pmatrix}, \qquad j = 1, \ldots, k,$$

  where $s_1 \geq \cdots \geq s_k > 0$ are nonzero singular values of A.
- The singular values of a skew-symmetric matrix $A \in M_n$ occur in pairs.
- Two symmetric (skew-symmetric) matrices are unitarily congruent if and only if they have the same singular values.

Proof. Suppose $A \in M_n$ is symmetric. Let $\mathbf{x} \in \mathbb{C}^n$ be a unit vector so that $\mathbf{x}^t A\mathbf{x}$ is real and maximum, and let $U \in M_n$ be unitary with $\mathbf{x}$ as the first column. Show that $U^t AU = [s_1] \oplus A_1$. Then use induction.

Suppose $A \in M_n$ is skew-symmetric. Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ be orthonormal pairs such that $\mathbf{x}^t A\mathbf{y}$ is real and maximum, and $U \in M_n$ be unitary with $\mathbf{x}, \mathbf{y}$ as the first two columns. Show that $U^t AU = \begin{pmatrix} 0 & s_1 \\ -s_1 & 0 \end{pmatrix} \oplus A_1$. Then use induction. □

2.6 Remarks on real matrices

Theorem 2.6.1 *If $A \in M_n(\mathbb{R})$, then A is orthogonally similar to block triangular matrix (A_{ij}) such that A_{jj} is either 1×1 or of the forms $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.*

- *The matrix is normal, i.e., $AA^t = A^tA$, if and only if all the off-diagonal blocks are zero.*
- *The matrix A orthogonal, i.e., $AA^t = I_n$, if and only if the 1×1 diagonal blocks have the form $[1]$ or $[-1]$, and the entries in the 2×2 diagonal blocks satisfy $a^2 + b^2 = 1$.*
- *The matrix A is* **symmetric**, *i.e., $A = A^t$, if and only if it is orthogonally similar to a real diagonal matrices.*
- *The matrix A is* **symmetric** *and satisfies $x^tAx \geq 0$ for all $x \in \mathbb{R}^n$ if and only if it is orthogonally similar to a nonnegative diagonal matrices.*

Remark 2.6.2 *Let $A \in M_n(\mathbb{R})$. Then $A = S + K$ where $S = (A + A^t)/2$ is symmetric and $K = (A - A^t)/2$ is skew-symmetric, i.e., $K^t = -K$.*

- *Note that $x^tKx = 0$ for all $x \in \mathbb{R}^n$.*
- *Clearly, $x^tAx \in \mathbb{R}$ for all real vectors $x \in \mathbb{R}^n$, and the condition does not imply that A is symmetric as in the complex Hermitian case.*
- *The matrix A satisfies $x^tAx \geq 0$ for all if and only if $(A + A^t)/2$ has only nonnegative eigenvalues. The condition does not automatically imply that A is symmetric as in the complex Hermitian case.*
- *Every skew-symmetric matrix $K \in M_n(\mathbb{R})$ is orthogonally similar to 0_{2k} and*

$$\begin{pmatrix} 0 & s_j \\ -s_j & 0 \end{pmatrix}, \qquad j = 1, \ldots, k,$$

  where $s_1 \geq \cdots \geq s_k > 0$ are nonzero singular values of A.

3 Eigenvalues and singular values inequalities

We study inequalities relating the eigenvalues, diagonal elements, singular values of matrices in this chapter.

For a Hermitian matrix A, let $\lambda(A) = (\lambda_1(A), \dots, \lambda_n(A))$ be the vector of eigenvalues of A with entries arranged in descending order. Also, we will denote by $s(A) = (s_1(A), \dots, s_n(A))$ the singular values of a matrix $A \in M_{m,n}$. For two Hermitian matrices, we write $A \geq B$ if $A - B$ is positive semidefinite.

3.1 Diagonal entries and eigenvalues of a Hermitian matrix

Theorem *Let $A = (a_{ij}) \in M_n$ be Hermitian with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Then for any $1 \leq k < n$, $a_{11} + \cdots + a_{kk} \leq \lambda_1 + \cdots + \lambda_k$. The equality holds if and only if $A = A_{11} \oplus A_{22}$ so that A_{11} has eigenvalues $\lambda_1, \dots, \lambda_k$.*

Remark The above result will give us what we needed, and we can put the majorization result as a related result for real vectors.

Lemma 3.1.1 (Rayleigh principle) *Let $A \in M_n$ be Hermitian. Then for any unit vector $\mathbf{x} \in \mathbb{C}^n$,*

$$\lambda_1(A) \geq \mathbf{x}^* A \mathbf{x} \geq \lambda_n(A).$$

The equalities hold at unit eigenvectors corresponding to the largest and smallest eigenvalues of A, respectively.

Proof. Done in homework problem. □

If we take $\mathbf{x} = e_j$, we see that every diagonal entry of a Hermitian matrix A lies between $\lambda_1(A)$ and $\lambda_n(A)$.

We can say more in the following. To do that we need the notion of **majorization** and **doubly stochastic matrices**.

A matrix $D = (d_{ij}) \in M_n$ is doubly stochastic if $d_{ij} \geq 0$ and all the row sums and column sums of D equal 1.

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. We say that $\mathbf{x}$ is weakly majorized by $\mathbf{y}$, denoted by $\mathbf{x} \prec_w \mathbf{y}$ if the sum of the k largest entries of $\mathbf{x}$ is not larger than that of $\mathbf{y}$ for $k = 1, \dots, n$; in addition, if the sum of the entries of $\mathbf{x}$ and $\mathbf{y}$, we say that $\mathbf{x}$ is majorized by $\mathbf{y}$, denoted by $\mathbf{x} \prec \mathbf{y}$. We say that $\mathbf{x}$ is obtained from $\mathbf{y}$ be a pinching if $\mathbf{x}$ is obtained from $\mathbf{y}$ by changing (y_i, y_j) to $(y_i - \delta, y_j + \delta)$ for two of the entries $y_i > j_j$ of y and some $\delta \in (0, y_i - y_j)$.

Theorem 3.1.2 *Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ with $n \geq 2$. The following conditions are equivalent.*

(a) $\mathbf{x} \prec \mathbf{y}$.

(b) *There are vectors* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ *with* $k < n$, $\mathbf{x}_1 = \mathbf{y}$, $\mathbf{x}_k = \mathbf{x}$, *such that each* $\mathbf{x}_j$ *is obtained from* $\mathbf{x}_{j-1}$ *by pinching two of its entries.*

(c) $\mathbf{x} = D\mathbf{y}$ *for some doubly stochastic matrix.*

Proof. Note that the conditions do not change if we replace $(\mathbf{x}, \mathbf{y})$ by $(P\mathbf{x}, Q\mathbf{y})$ for any permutation matrices P, Q. We may make these changes in our proof.

(c) $\Rightarrow$ (a). We may assume that $\mathbf{x} = (x_1, \ldots, x_n)^t$ and $\mathbf{y} = (y_1, \ldots, y_n)^t$ with entries in descending order. Suppose $\mathbf{x} = D\mathbf{y}$ for a doubly stochastic matrix $D = (d_{ij})$. Let $\mathbf{v}_k = (e_1 + \cdots + e_k)$ and $\mathbf{v}_k^t D = (c_1, \ldots, c_n)$. Then $0 \le c_j \le 1$ and $\sum_{j=1}^n c_j = k$. So,

$$\begin{aligned}\sum_{j=1}^k x_j &= \mathbf{v}_k^t D y = c_1 y_1 + \cdots + c_n y_n \\ &\le c_1 y_1 + c_k y_k + [(1 - c_1) + \cdots + [1 - c_k)] y_k \le y_1 + \cdots + y_k.\end{aligned}$$

Clearly, the equality holds if $k = n$.

(a) $\Rightarrow$ (b). We prove the result by induction on n. If $n = 2$, the result is clear. Suppose the result holds for vectors of length less than n. Assume $\mathbf{x} = (x_1, \ldots, x_n)^t$ and $\mathbf{y} = (y_1, \ldots, y_n)^t$ has entries arranged in descending order, and $\mathbf{x} \prec \mathbf{y}$. Let k be the maximum integer such that $y_k \ge x_1$. If $k = n$, then for $S = \sum_{j=1}^n x_j = \sum_{j=1}^n y_j$,

$$y_n \ge x_1 \ge \cdots x_n \ge = S - \sum_{j=1}^{n-1} x_j \ge S - \sum_{j=1}^{n-1} y_j = y_n$$

so that $x = \cdots = x_n = y_1 = \cdots = y_n$. So, $\mathbf{x} = \mathbf{x}_1 = \mathbf{y}$. Suppose $k < n$ and $y_k \ge x_1 > y_{k+1}$. Then we can replace (y_k, y_{k+1}) by $(\tilde{y}_k, \tilde{y}_{k+1}) = (x_1, y_k + y_{k+1} - x_1)$. Then removing x_1 from $\mathbf{x}$ and removing $\tilde{y}_k$ in $\mathbf{x}_1$ will yield the vectors $\tilde{\mathbf{x}} = (x_2, \ldots, x_n)^t$ and $\tilde{\mathbf{y}} = (y_1, \ldots, y_{k-1}, \tilde{y}_{k+1}, \ldots, y_n)^t$ in $\mathbb{R}^{n-1}$ with entries arranged in descending order. We will show that $\tilde{\mathbf{x}} \prec \tilde{\mathbf{y}}$. The result will then follows by induction. Now, if $\ell \le k$, then

$$x_2 + \cdots + x_\ell \le x_1 + \cdots + x_{\ell-1} \le y_1 + \cdots + y_{\ell-1};$$

if $\ell > k$, then

$$x_2 + \cdots + x_\ell \le (y_1 + \cdots + y_\ell) - x_1 = y_1 + \cdots + y_{k-1} + \tilde{y}_{k+1} + y_{k+1} + \cdots + y_\ell$$

with equality when $\ell = n$. The results follows.

(b) $\Rightarrow$ (c). If $\mathbf{x}_j$ is obtained from $\mathbf{x}_{j-1}$ by pinching the pth and qth entries. Then there is a doubly stochastic matrix P_j obtained from I by changing the submatrix in rows and columns p, q by

$$\begin{pmatrix} t_j & 1 - t_j \\ 1 - t_j & t_j \end{pmatrix}$$

for some $t_j \in (0,1)$. Then $\mathbf{x} = D\mathbf{y}$ for $D = P_k \cdots P_1$, which is doubly stochastic. $\square$

Theorem 3.1.3 *Let $\mathbf{d}, \mathbf{a} \in \mathbb{R}^n$. The following are equivalent.*

(a) *There is a complex Hermitian (real symmetric) $A \in M_n$ with entries of $\mathbf{a}$ as eigenvalues and entries of $\mathbf{d}$ as diagonal entries.*

(b) *The vectors satisfy $\mathbf{d} \prec \mathbf{a}$.*

Proof. Let $A = UDU^*$ such that $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n))$. Suppose $A = (a_{ij})$ and $U = (u_{ij})$. Then $a_{jj} = \sum_{i=1}^n \lambda_i |u_{ji}|^2$. Because $(|u_{ji}|^2)$ is doubly stochastic. So, $(a_{11}, \ldots, a_{nn}) \prec (\lambda_1, \ldots, \lambda_n)$.

We prove the converse by induction on n. Suppose $(d_1, \ldots, d_n) \prec (\lambda_1, \ldots, \lambda_n)$. If $n = 2$, let $d_1 = \lambda_1 \cos^2\theta + \lambda_2 \sin^2\theta$ so that

$$(a_{ij}) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

has diagonal entries d_1, d_2.

Suppose $n > 2$. Choose the maximum k such that $\lambda_k \ge d_1$. If $\lambda_n = d_1$, then for $S = \sum_{j=1}^n d_j = \sum_{j=1}^n \lambda_j$ we have

$$\lambda_n \ge d_1 \ge \cdots \ge d_n = S - \sum_{j=1}^{n-1} d_j \ge S - \sum_{j=1}^{n-1} \lambda_j = \lambda_n.$$

Thus, $\lambda_n = d_1 = \cdots = d_n = S/n = \sum_{j=1}^n \lambda_j / n$ implies that $\lambda_1 = \cdots = \lambda_n$. Hence, $A = \lambda_n I$ is the required matrix. Suppose $k < n$. Then there is $A_1 = A_1^t \in M_2(\mathbb{R})$ with diagonal entries $d_1, \lambda_k + \lambda_{k+1} - d_1$ and eigenvalues λ_j, λ_{j+1}. Consider $A = A_1 \oplus D$ with $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_{k-1}, \lambda_{k+2}, \ldots, \lambda_n)$. As shown in the proof of Theorem 3.1.3, if $\tilde{\lambda}_{k+1} = \lambda_k + \lambda_{k+1} - d_1$, then

$$(d_2, \ldots, d_n) \prec (\tilde{\lambda}_{k+1}, \lambda_1, \ldots, \lambda_{k-1}, \lambda_{k+2}, \ldots, \lambda_n).$$

By induction assumption, there is a unitary $U \in M_{n-1}$ such that

$$U([\tilde{\lambda}_k] \oplus D)U^* \in M_{n-1}$$

has diagonal entries $d_2, \ldots, d_n$. Thus, $A = ([1] \oplus U)(A_1 \oplus D)([1] \oplus U^*)$ has the desired eigenvalues and diagonal entries. $\square$

3.2 Max-Min and Min-Max characterization of eigenvalues

In this subsection, we give a Max-Min and Min-Max characterization of eigenvalues of a Hermitian matrix.

Lemma 3.2.1 *Let V_1 and V_2 be subspaces of $\mathbb{C}^n$ such that $\dim(V_1)+\dim(V_2) > n$, then $V_1 \cap V_2 \neq \{0\}$.*

Proof. Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ and $\{\mathbf{v}_1, \ldots, \mathbf{v}_q\}$ be bases for V_1 and V_2. Then $p+q > n$ and the linear system $[\mathbf{u}_1 \cdots \mathbf{u}_p \mathbf{v}_1 \cdots \mathbf{v}_q]\mathbf{x} = \mathbf{0} \in \mathbb{C}^n$ has a non-trivial solution $\mathbf{x} = (x_1, \ldots, x_p, y_1, \ldots, y_q)^t$. Note that not all $x_1, \ldots, x_p$ are zero, else $y_1\mathbf{v}_1 + \cdots + y_q\mathbf{v}_1 = 0$ implies $y_j = 0$ for all j. Thus, $\mathbf{v} = x_1\mathbf{u}_1 + \cdots + x_p\mathbf{u}_p = -(y_1\mathbf{v}_1 + \cdots + y_q\mathbf{v}_q)$ is a nonzero vector in $\mathbf{V}_1 \cap \mathbf{V}_2$. □

Theorem 3.2.2 *Let $A \in M_n$ be Hermitian. Then for $1 \leq k \leq n$,*

$$\begin{aligned}\lambda_k(A) &= \max\{\lambda_k(X^*AX) : X \in M_{n,k}, X^*X = I_k\} \\ &= \min\{\lambda_1(Y^*AY) : Y \in M_{n,n-k+1}, Y^*Y = I_{n-k+1}\}.\end{aligned}$$

Proof. Suppose $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is a family of orthonormal eigenvectors of A corresponding to the eigenvalues $\lambda_1(A), \ldots, \lambda_n(A)$. Let $X = [\mathbf{u}_1 \cdots \mathbf{u}_k]$. Then $X^*AX = \operatorname{diag}(\lambda_1(A), \ldots, \lambda_k(A))$ so that

$$\lambda_k(A) \leq \max\{\lambda_k(X^*AX) : X \in M_{n,k}, X^*X = I_k\}.$$

Conversely, suppose X has orthonomals column $\mathbf{x}_1, \ldots, \mathbf{x}_k$ spanning a subspace V_1. Let $\mathbf{u}_k, \ldots, \mathbf{u}_n$ span a subspace V_2 of dimension $n-k+1$. Then there is a unit vector $\mathbf{v} = \sum_{j=1}^k x_j\mathbf{x}_j = \sum_{j=k}^n y_j\mathbf{u}_j$. Let $\mathbf{x} = (x_1, \ldots, x_k)^t, \mathbf{y} = (y_k, \ldots, y_{n-k})^t$, $Y = [\mathbf{u}_k \ \ldots \ \mathbf{u}_{k+1}]$. Then $\mathbf{v} = X\mathbf{x} = Y\mathbf{y}$ so that $Y^*AY = \operatorname{diag}(\lambda_k(A), \ldots, \lambda_n(A))$. By Rayleigh principle,

$$\lambda_k(X^*AX) \leq \mathbf{x}^tX^*AX\mathbf{x} = \mathbf{y}^tY^*AY\mathbf{y} \leq \lambda_k(A).$$

□

3.3 Change of eigenvalues under perturbation

Theorem 3.3.1 *Suppose $A, B \in M_n$ are Hermitian such that $A \geq B$. Then $\lambda_k(A) \geq \lambda_k(B)$ for all $k = 1, \ldots, n$.*

Proof. Let $A = B + P$, where P is positive semidefinite. Suppose $k \in \{1, \ldots, n\}$. There is $Y \in M_{n,k}$ with $Y^*Y = I_k$ such that

$$\lambda_k(B) = \lambda_k(Y^*BY) = \max\{\lambda_k(X^*BX) : X \in M_{m,n}, X^*X = I_k\}.$$

Let $\mathbf{y} \in \mathbb{C}^k$ be a unit eigenvector of Y^*AY corresponding to $\lambda_k(X^*AX)$. Then

$$\begin{aligned}\lambda_k(A) &= \max\{\lambda_k(X^*AX) : X \in M_{m,n}, X^*X = I_k\} \\ &\geq \lambda_k(Y^*AY) = \mathbf{y}^*Y^*(B+P)Y\mathbf{y} = \mathbf{y}^*Y^*BY\mathbf{y} + \mathbf{y}^*Y^*PY\mathbf{y} \\ &\geq \mathbf{y}^*Y^*BY\mathbf{y} \geq \lambda_k(Y^*BY) = \lambda_k(B).\end{aligned}$$

□

Theorem 3.3.2 (Lidskii) *Let $A, B, C = A + B \in M_n$ be Hermitian matrices with eigenvalues $a_1 \ge \cdots \ge a_n$, $b_1 \ge \cdots \ge b_n$, $c_1 \ge \cdots \ge c_n$, respectively. Then $\sum_{j=1}^n a_j + \sum_{j=1}^n b_j = \sum_{j=1}^n c_j$ and for any $1 \le r_1 < \cdots < r_k \le n$,*

$$\sum_{j=1}^k b_{n-j+1} \le \sum_{j=1}^k (c_{r_j} - a_{r_j}) \le \sum_{j=1}^k b_j.$$

Proof. Suppose $1 \le r_1 < \cdots < r_k \le n$. We want to show $\sum_{j=1}^k (c_{r_j} - a_{r_j}) \le \sum_{j=1}^k b_j$. Replace B by $B - b_k I$. Then each eieganvelue of B and each eigenvalue of $C = A + B$ will be changed by $-b_k$. So, it will not affect the inequalities. Suppose $B = \sum_{j=1}^n b_j \mathbf{x}_j \mathbf{x}_j^*$. Let $B_+ = \sum_{j=1}^k b_j \mathbf{x}_j \mathbf{x}_j^*$. Then

$$\begin{aligned} \sum_{j=1}^k (c_{r_j} - a_{r_j}) &\le \sum_{j=1}^k (\lambda_{r_j}(A + B_+) - \lambda_{r_j}(A)) \quad \text{because } \lambda_j(A+B) \le \lambda_j(A+B_+) \text{ for all } j \\ &\le \sum_{j=1}^n (\lambda_j(A + B_+) - \lambda_j(A)) \quad \text{because } \lambda_j(A) \le \lambda_j(A+B_+) \text{ for all } j \\ &= \operatorname{tr}(A + B_+) - \operatorname{tr}(A) = \sum_{j=1}^k \lambda_j(B_+) = \sum_{j=1}^k b_j. \end{aligned}$$

Replacing (A, B, C) by $(-A, -B, -C)$, we get the other inequalities. □

Lemma 3.3.3 *Suppose $A \in M_{m,n}$ has nonzero singular values $s_1 \ge \cdots \ge s_k$. Then $\begin{pmatrix} 0_m & A \\ A^* & 0_n \end{pmatrix}$ has nonzero eigenvalues $\pm s_1, \ldots, \pm s_k$.*

Theorem 3.3.4 *Let $A, B, C \in M_{m,n}$ with singular values $a_1 \ge \cdots \ge a_n, b_1 \ge \cdots \ge b_n$ and $c_1, \ldots, c_n$, respectively. Then for any $1 \le j_1 < \cdots < j_k \le n$, we have*

$$\sum_{j=1}^k (c_{r_j} - a_{r_j}) \le \sum_{j=1}^k b_j.$$

3.4 Eigenvalues of principal submatrices

Theorem 3.4.1 *There is a positive matrix $C = \begin{pmatrix} A & * \\ * & B \end{pmatrix}$ with $A \in M_k$ so that A, B, C have eigenvalues $a_1 \ge \cdots \ge a_k$, $b_1 \ge \cdots \ge b_{n-k}$ and $c_1 \ge \cdots \ge c_n$, respectively, if and only if there are positive semi-definite matrices $\tilde{A}, \tilde{B}, \tilde{C} = \tilde{A} + \tilde{B}$ with eigenvalues $a_1 \ge \cdots \ge a_k \ge 0 = a_{k+1} = \cdots = a_n$, $b_1 \ge \cdots \ge b_{n-k} \ge 0 = b_{n-k+1} = \cdots = b_n$, and $c_1 \ge \cdots \ge c_n$.*

Consequently, for any $1 \le j_1 < \cdots < j_k \le n$, we have $\sum_{j=1}^k (c_{r_j} - a_{r_j}) \le \sum_{j=1}^k b_j$.

Proof. To prove the necessity, let $C = \hat{C}^*\hat{C}$ with $\hat{C} = [C_1\ C_2] \in M_n$ with $C_1 \in M_{n,k}$. Then $A = C_1^*C_1$ has eigenvalues $a_1, \ldots, a_k$, and $B = C_2^*C_2$ has eigenvalues $b_1, \ldots, b_{n-k}$. Now, $\tilde{C} = \hat{C}\hat{C}^* = C_1C_1^* + C_2C_2^*$ also eigenvalues $c_1, \ldots, c_n$, and $\tilde{A} = C_1C_1^*, \tilde{B} = C_2C_2^*$ have the desired eigenvalues.

Conversely, suppose the $\tilde{A}, \tilde{B}, \tilde{C}$ have the said eigenvalues. Let $\tilde{A} = C_1C_1^*$, $\tilde{B} = C_2C_2^*$ for some $C_1 \in M_{n,k}, C_2 \in M_{n,n-k}$. Then $C = [C_1\ C_2]^* = [C_1\ C_2]$ have the desired principal submatrices. □

By the above theorem, one can apply the inequalities governing the eigenvalues of $\tilde{A}, \tilde{B}, \tilde{C} = \tilde{A} + \tilde{B}$ to deduce inequalities relating the eigenvalues of a positive semidefinite matrix C and its complementary principal submatrices. One can also consider general Hermitian matrix by studying $C - \lambda_n(C)I$.

Theorem 3.4.2 *There is a Hermitian (real symmetric) matrix $C \in M_n$ with principal submatrix $A \in M_m$ such that C and A have eigenvalues $c_1 \geq \cdots \geq c_n$ and $a_1 \geq \cdots \geq a_m$, respectively, if and only if*

$$c_j \geq a_j \quad \text{and} \quad a_{m-j+1} \geq c_{n-j+1}, \qquad j = 1, \ldots, m.$$

Proof. To prove the necessity, we may replace C by $C - \lambda_n(C)I$ and assume that C is positive semidefinite. Then by the previous theorem,

$$c_j - a_j \geq b_{n-m} \geq 0, \qquad j = 1, \ldots, m.$$

Applying the argument to $-C$, we get the conclusion.

To prove the sufficiency, we will construct $C - c_nI$ with principal submatrix $A - I_m$. Thus, we may assume that all the eigenvalues involved are nonnegative.

We prove the converse by induction on $n - m \in \{1, \ldots, n-1\}$. Suppose $n - m = 1$.

We need only address the case $\mu_j \in (\lambda_{j+1}, \lambda_j)$ for $j = 1, \ldots, n-1$, since the general case $\mu_j \in [\lambda_{j+1}, \lambda_j]$ follows by a continuity argument. Alternatively, we can take away the pairs of $c_j = a_j$ or $a_j = c_{j+1}$ to get a smaller set of numbers that still satisfy the interlacing inequalities and apply the following arguments.

We will show how to choose a real orthogonal matrix Q such that $C = Q^t \mathrm{diag}(\lambda_1, \ldots, \lambda_n)Q$ has the leading principal submatrix $A \in M_{n-1}$ with eigenvalues $a_1 \geq \cdots \geq a_{n-1}$. To this end, let Q have last column $u = (u_1, \ldots, u_n)^t$. By the adjoint formula for the inverse

$$[(zI - C)^{-1}]_{nn} = \frac{\det(zI_{n-1} - A))}{\det(I - C)} = \frac{\prod_{j=1}^{n-1}(z - a_j)}{\prod_{j=1}^{n}(z - c_j)},$$

but we also have the expression

$$(zI - A)^{-1}_{nn} = u^t(zI - \mathrm{diag}(\lambda_1, \ldots, \lambda_n))^{-1}u = \sum_{i=1}^{n} \frac{u_i^2}{(z - \lambda_i)}.$$

Equating these two, we see that $A(n)$ has characteristic polynomial $\prod_{i=1}^{n-1}(z-\mu_i)$ if and only if

$$\sum_{i=1}^{n} u_i^2 \prod_{j\neq i}(z-a_i) = \prod_{i=1}^{n-1}(z-c_i).$$

Both sides of this expression are polynomials of degree $n-1$ so they are identical if and only if they agree at the n distinct points $a_1,\ldots,c_n$, or equivalently,

$$u_k^2 = \frac{\prod_{j=1}^{n-1}(c_k-a_j)}{\prod_{j\neq k}(c_k-c_j)} \equiv w_k, \qquad k=1,\ldots,n.$$

Since $(c_k-a_j)/(c_k-c_j)>0$ for all $k\neq j$, we see that $w_k>0$. Thus if we take $u_k=\sqrt{w_k}$ then A has eigenvalues $a_1,\ldots,a_{n-1}$.

Now, suppose $m<n-1$. Let

$$\tilde{c}_j = \begin{cases} \max\{c_{j+1},a_j\} & 1\leq j\leq m,\\ \min\{c_j,a_{m-n+j+1}\} & m<j<n. \end{cases}$$

Then

$$c_1\geq \tilde{c}_1\geq c_2\geq\cdots\geq c_{n-1}\geq\tilde{c}_{n-1}\geq c_n,$$

and

$$\tilde{c}_j\geq a_j\geq \tilde{c}_{n-m-1+j}, \quad j=1,\ldots,m.$$

By the induction assumption, we can construct a Hermitian $\tilde{C}\in M_{n-1}$ with eigenvalues $\tilde{c}_1\geq\cdots\geq\tilde{c}_{n-1}$, whose $m\times m$ leading principal submatrix has eigenvalues $a_1\geq\cdots\geq a_m$, and $\tilde{C}$ is the leading principal submatrix of the real symmetric matrix $C\in M_n$ such that C has eigenvalues $c_1\geq\cdots\geq c_n$. □

3.5 Eigenvalues and Singular values

Theorem 3.5.1 *Let* $A=\begin{pmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}\in M_n$ *with* $A_{11}\in M_k$. *Then* $|\det(A_{11})|\leq\prod_{j=1}^k s_j(A)$. *The equality holds if and only if* $A=A_{11}\oplus A_{22}$ *such that* A_{11} *has singular values* $s_1(A),\ldots,s_k(A)$.

Proof. Let $\mathcal{S}(s_1,\ldots,s_n)$ be the set of matrices in M_n with singular values $s_1\geq\cdots\geq s_n$. Suppose $A=\begin{pmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}\in\mathcal{S}(s_1,\ldots,s_n)$ with $A_{11}\in M_k$ such that $|\det(A_{11})|$ attains the maximum value. We show that $A=A_{11}\oplus A_{22}$ and A_{11} has singular values $s_1\geq\cdots\geq s_k$.

Suppose $U,V\in M_k$ are such that $U^*A_{11}V=\operatorname{diag}(\xi_1,\ldots,\xi_k)$ with $\xi_1\geq\cdots\geq\xi_k\geq 0$. We may replace A by $(U^*\oplus I_{n-k})A(V\oplus I_{n-1})$ and assume that $A_{11}=\operatorname{diag}(\xi_1,\ldots,\xi_k)$.

Let $A = (a_{ij})$. We show that $A_{21} = 0$ as follows. Suppose there is a nonzero entry a_{s1} with $k < s \le n$. Then there is a unitary $X \in M_2$ such that $X \begin{pmatrix} a_{11} & a_{s1} \\ a_{1s} & a_{ss} \end{pmatrix}$ has $(1,1)$ entry equal to

$$\hat{\xi}_1 = \{|a_{11}|^2 + |a_{s1}|^2\}^{1/2} = \{\xi_1^2 + |a_{s1}|^2\}^{1/2} > \xi_1.$$

Let $\hat{X} \in M_n$ be obtained from I_n by replacing the submatrix in rows and columns $1, j$ by X. Then the leading $k \times k$ submatrix of $\hat{X}A$ is obtained from that of A by changing its first row from $(\xi_1, 0, \dots, 0)$ to $(\hat{\xi}_1, *, \cdots, *)$, and has determinant $\hat{\xi}_1\xi_2 \cdots \xi_k > \xi_1 \cdots \xi_k = \det(A_{11})$, contradicting the fact that $|\det(A_{11})|$ attains the maximum value. Thus, the first column of A_{21} is zero.

Next, suppose that there is $a_{s2} \neq 0$ for some $k < s \le n$. Then there is a unitary $X \in M_2$ such that $X \begin{pmatrix} a_{22} & a_{s2} \\ a_{2s} & a_{ss} \end{pmatrix}$ has $(1,1)$ entry equal to

$$\hat{\xi}_2 = \{|a_{22}|^2 + |a_{s2}|^2\}^{1/2} = \{\xi_2^2 + |a_{s2}|^2\}^{1/2} > \xi_2.$$

Then the leading $k \times k$ submatrix of $\hat{X}A$ is obtained from that of A by changing its first row from $(0, \xi_2, 0, \dots, 0)$ to $(0, \hat{\xi}_2, *, \cdots, *)$, and has determinant $\xi_1\hat{\xi}_2 \cdots \xi_k > \xi_1 \cdots \xi_k = \det(A_{11})$, which is a contradiction. So, the second column of A_{21} is zero. Repeating this argument, we see that $A_{21} = 0$.

Now, the leading $k \times k$ submatrix of $A^t \in \mathcal{S}(s_1, \dots, s_n)$ also attains the maximum. Applying the above argument, we see that $A_{12}^t = 0$. So, $A = A_{11} \oplus A_{22}$.

Let $\hat{U}, \hat{V} \in M_{n-k}$ be unitary such that $\hat{U}^* A_{22} \hat{V} = \operatorname{diag}(\xi_{k+1}, \dots, \xi_n)$. We may replace A by $(I_k \oplus \hat{U}^*)A(I_k \oplus \hat{V})$ so that $A = \operatorname{diag}(\xi_1, \dots, \xi_n)$. Clearly, $\xi_k \ge \xi_{k+1}$. Otherwise, we may interchange kth and $(k+1)$st rows and also the columns so that the leading $k \times k$ submatrix of the resulting matrix becomes $\operatorname{diag}(\xi_1, \dots, \xi_{k-1}, \xi_{k+1})$ with determinant larger than $\det(A_{11})$. So, $\xi_1, \dots, \xi_k$ are the k largest singular values of A. □

Theorem 3.5.2 *Let $a_1, \dots, a_n$ be complex numbers be such that $|a_1| \ge \cdots \ge |a_n|$ and $s_1 \ge \cdots \ge s_n \ge 0$. Then there is $A \in M_n$ with eigenvalues $a_1, \dots, a_n$ and singular values $s_1, \cdots, s_n$ if and only if*

$$\prod_{j=1}^{n} |a_j| = \prod_{j=1}^{n} s_j, \quad \text{and} \quad \prod_{j=1}^{k} |a_j| \le \prod_{j=1}^{k} s_j \quad \text{for } \ j = 1, \dots, n-1.$$

Proof. Suppose A has eigenvalues $a_1, \dots, a_n$ and singular values $s_1 \ge \cdots \ge s_n \ge 0$. We may apply a unitary similarity to A and assume that A is in upper triangular form with diagonal entries $a_1, \dots, a_n$. By the previous theorem, if A_k is the leading $k \times k$ submatrix of A, then $|a_1 \cdots a_k| = |\det(A_k)| \le \prod_{j=1}^k s_k$ for $k = 1, \dots, n-1$, and $|\det(A)| = |a_1 \cdots a_n| = s_1 \cdots s_n$.

To prove the converse, suppose the asserted inequalities and equality on $a_1, \dots, a_n$ and $s_1, \dots, s_n$ hold. We show by induction that there is an upper triangular matrix $A = (a_{ij})$ with singular values

$s_1 \geq \cdots \geq s_n$ and diagonal values $|a_1|, \dots, |a_n|$ (in a certain order). Then there will be a diagonal unitary matrix D such that DA has the desired eigenvalues and singular values. For notation simplicity, we assume $a_j = |a_j|$ in the following.

Suppose $n = 2$. Then $a_1 \leq s_1$, and $a_1 a_2 = s_1 s_2$ so that $a_1 \geq a_2 \geq s_2$. Consider

$$A(\theta, \phi) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} s_1 & \\ & s_2 \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}.$$

There is $\phi \in [0, \pi/2]$ such that the $(s_1 \cos\phi, s_2 \sin\phi)^t$ has norm $a_1 \in [s_2, s_1]$. Then we can find $\theta \in [0, \pi/2]$ such that $(\cos\theta, \sin\theta)(s_1 \cos\phi, s_2 \sin\phi) = a_1$. Thus, the first column of $A(\theta, \phi)$ equals $(a_1, 0)^t$, and $A(\theta, \phi)$ has the desired eigenvalues and singular values.

Suppose the result holds for matrices of size at most $n - 1 \geq 2$. Consider $(a_1, \dots, a_n)$ and $(s_1, \dots, s_n)$ satisfying the product equality and inequalities.

If $a_1 = 0$, then $s_n = 0$ and $A = s_1 E_{12} + \cdots + s_{n-1} E_{n-1,n}$ has the desired eigenvalues and singular values.

Suppose $a_1 > 0$. Let k be the maximum integer such that $s_k \geq a_1$. Then there is $A_1 = \begin{pmatrix} a_1 & * \\ 0 & \tilde{s}_{k+1} \end{pmatrix}$ with $\tilde{s}_{k+1} = s_k s_{k+1}/a_k \in [s_{k-1}, s_{k+1}]$. Let

$$(\tilde{s}_1, \dots, \tilde{s}_{n-1}) = (s_1, \dots, s_{k-1}, \tilde{s}_{k+1}, s_{k+2}, \dots, s_n).$$

We claim that $(a_2, \dots, a_n)$ and $(\tilde{s}_1, \dots, \tilde{s}_{n-1})$ satisfy the product equality and inequalities. First, $\prod_{j=2}^n a_j = \prod_{j=1}^n s_j / a_1 = \prod_{j=1}^{n-1} \tilde{s}_j$. For $\ell < k$,

$$\prod_{j=2}^{\ell} a_j \leq \prod_{j=1}^{\ell-1} a_j \leq \prod_{j=1}^{\ell-1} s_j = \prod_{j=1}^{\ell-1} \tilde{s}_j.$$

For $\ell \geq k + 1$,

$$\prod_{j=2}^{\ell} a_j \leq \prod_{j=1}^{\ell} s_j / a_1 = \prod_{j=1}^{\ell-1} \tilde{s}_j.$$

So, there is $A_2 \in U\tilde{D}V^*$ in triangular form with diagonal entries $a_2, \dots, a_n$, where $U, V \in M_{n-1}$ are unitary, and $\tilde{D} = \operatorname{diag}(\tilde{s}_1, \dots, \tilde{s}_n)$. Let

$$A = \begin{pmatrix} 1 & \\ & U \end{pmatrix} \begin{pmatrix} A_0 & \\ & \tilde{D} \end{pmatrix} \begin{pmatrix} 1 & \\ & V^* \end{pmatrix}$$

is in upper triangular form with diagonal entries $a_k, a_1, \dots, a_{k-1}, a_{k+1}, \dots, a_n$ and singular values $s_1, \dots, s_n$ as desired. $\square$

3.6 Diagonal entries and singular values

Theorem 3.6.1 *Let $A \in M_n$ have diagonal entries $d_1, \dots, d_n$ such that $|d_1| \ge \cdots \ge |d_n|$ and singular values $s_1 \ge \cdots \ge s_n$.*

(a) *For any $1 \le k \le n$, we have $\sum_{j=1}^k |d_j| \le \sum_{j=1}^k s_j$. The equality holds if and only if there is a diagonal unitary matrix D such that $DA = A_{11} \oplus A_{22}$ such that A_{11} is positive semidefinite with eigenvalues $s_1 \ge \cdots \ge s_k$.*

(b) *We have $\sum_{j=1}^{n-1} |d_j| - |d_n| \le \sum_{j=1}^{n-1} s_j - s_n$. The equality holds if and only if there is a diagonal unitary matrix D such that $DA = (a_{ij})$ is Hermitian with eigenvalues $s_1, \dots, s_{n-1}, -s_n$ and $a_{nn} \le 0$.*

Proof. (a) Let $\mathcal{S}(s_1, \dots, s_n)$ be the set of matrices in M_n with singular values $s_1 \ge \cdots \ge s_n$. Suppose $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \mathcal{S}(s_1, \dots, s_n)$ with $A_{11} \in M_k$ such that $|a_{11}| + \cdots + |a_{kk}|$ attains the maximum value. We may replace A by DA by a suitable diagonal unitary $D \in M_n$ and assume that $a_{jj} = |a_{jj}|$ for all $j = 1, \dots, n$. If $a_{ij} \ne 0$ for any $j > k \ge i$, then there is a unitary $X \in M_2$ such that $X \begin{pmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{pmatrix}$ has $(1,1)$ entry equal to

$$\tilde{a}_{ii} = \{|a_{ii}|^2 + |a_{ji}|^2\}^{1/2} > |a_{ii}|.$$

Let $\hat{X} \in M_n$ be obtained from I_n by replacing the submatrix in rows and columns i, j by X. Then diagonal entries of the leading $k \times k$ submatrix $\hat{A}_{11}$ of $\hat{X}A$ is obtained from that of A by changing its (i,i) entry a_{ii} to $\hat{a}_{ii}$ so that $\mathrm{tr}\, \hat{A}_{11} > \mathrm{tr}\, A_{11}$, which is a contradiction. So, $A_{12} = 0$. Applying the same argument to A^t, we see that $A_{12} = 0$. Now, A_{11} has singular values $\xi_1 \ge \cdots \ge \xi_k$. Then $A_{11} = PV$ for some positive semidefinite matrix P with eigenvalues $\xi_1, \dots, \xi_k$ and a unitary matrix $V \in M_k$. Suppose $V = U\hat{D}U^*$ for some diagonal unitary $\hat{D} \in M_k$ and unitary $U \in M_k$. Then

$$\mathrm{tr}\, A_{11} = \mathrm{tr}\,(PU\hat{D}U^*) = \mathrm{tr}\, U^*PU\hat{D} \le \mathrm{tr}\, U^*PU = \mathrm{tr}\, P,$$

where the equality holds if and only if $\hat{D} = I_k$, i.e., $A_{11} = P$ is positive semidefinite. In particular, we can choose $B = \mathrm{diag}\,(s_1, \dots, s_n)$ so that the sum of the k diagonal entries is $\sum_{j=1}^k s_j \ge \sum_{j=1}^k \xi_j = \mathrm{tr}\, A_{11}$. Thus, the eigenvalues of A_{11} must be $s_1, \dots, s_k$ as asserted.

(b) Let $A = (a_{ij}) \in \mathcal{S}(s_1, \dots, s_n)$ attains the maximum values $\sum_{j=1}^{n-1} |a_{jj}| - |a_{nn}|$. We may replace A by a diagonal unitary matrix and assume that $a_{ii} \ge 0$ for $j = 1, \dots, n-1$, and $a_{nn} \le 0$. Let $A_{11} \in M_{n-1}$ be the leading $(n-1) \times (n-1)$ principal submatrix of A. By part (a), we may assume that A_{11} is positive semidefinite so that its trace equals to the sum of its singular values. Otherwise, there are $U, V \in M_{n-1}$ such that $U^*A_{11}V = \mathrm{diag}\,(\xi_1, \dots, \xi_{n-1})$ with $\xi_1 + \cdots + \xi_{n-1} > \sum_{j=1}^{n-1} a_{jj}$.

As a result, $(U^* \oplus [1])A(V \oplus [1]) \in \mathcal{S}(s_1, \ldots, s_n)$ has diagonal entries $\hat{d}_1, \ldots, \hat{d}_{n-1}, a_{nn}$ such that

$$\sum_{j=1}^{n-1} \hat{d}_j - |a_{nn}| > \sum_{j=1}^{n-1} a_{jj} - |a_{nn}|,$$

which is a contradiction.

Next, for $j = 1, \ldots, n-1$, let $B_j = \begin{pmatrix} a_{jj} & a_{jn} \\ a_{nj} & a_{nn} \end{pmatrix}$. We show that $|a_{jj}| - |a_{nn}| = s_1(B_j) - s_2(B_j)$ and B_j is Hermitian in the following. Note that $s_1(B_1)^2 + s_2(B_j)^2 = |a_{jj}|^2 + |a_{jn}|^2 + |a_{nj}|^2 + |a_{nn}|^2$ and $s_1(B_j)s_2(B_j) = |a_{jj}a_{nn} - a_{jn}a_{nj}|$ so that $-a_{jj}a_{nn} = |a_{jj}a_{nn}| \geq s_1(B_j)s_2(B_j) - |a_{jn}a_{nj}|$. Hence,

$$\begin{aligned}
(|a_{jj}| - |a_{nn}|)^2 &= (a_{jj} + a_{nn})^2 = a_{jj}^2 + a_{nn}^2 + 2a_{jj}a_{nn} \\
&\leq s_1(B_j)^2 + s_2(B_j)^2 - (|a_{jk}|^2 + |a_{kj}|^2) - 2(s_1(B_j)s_2(B_j) - |a_{jn}a_{nj}|) \\
&= (s_1(B_j) - s_2(B_j))^2 - (|a_{jk}| - |a_{kj}|)^2 \\
&\leq (s_1(B_j) - s_2(B_j))^2.
\end{aligned}$$

Here the two inequalities become equalities if and only if $|a_{jk}| = |a_{kj}|$ and $|a_{jn}a_{nj}| = a_{jn}a_{nj}$, i.e., $a_{jn} = \bar{a}_{nj}$ and B_j is Hermitian.

By the above analysis, $|a_{jj}| - |a_{nn}| \leq s_1(B_j) - s_2(B_j)$. If the inequality is strict, there are unitary $X, Y \in M_2$ such that $X^* B_j Y = \operatorname{diag}(s_1(B_j), s_2(B_j))$. Let $\hat{X}$ be obtained from I_n by replacing the 2×2 submatrix in rows and columns j, n by X. Similarly, we can construct $\hat{Y}$. Then $\hat{X}, \hat{Y} \in M_n$ are unitary and $\hat{X}^* A \hat{Y}$ has diagonal entries $\hat{d}_1, \ldots, \hat{d}_n$ obtained from that of A by changing (a_{jj}, a_{nn}) to $(s_1(B_j), s_2(B_j))$. As a result,

$$\sum_{j=1}^{n-1} \hat{d}_j - |\hat{d}_n| > \sum_{j=1}^{n-1} a_{jj} - |a_{nn}|,$$

which is a contradiction. So, B_j is Hermitian for $j = 1, \ldots, n-1$. Hence, A is Hermitian, and

$$\operatorname{tr} A = a_{11} + \cdots + a_{nn} = a_{11} + \cdots + a_{n-1,n-1} - a_{nn}.$$

Suppose A has eigenvalues $\lambda_1, \ldots, \lambda_n$ with $|\lambda_j| = s_j(A)$ for $j = 1, \ldots, n$. Because $0 \geq a_{nn} \geq \lambda_n$, we see that $\operatorname{tr} A = \sum_{j=1} \lambda_j \leq \sum_{j=1}^{n-1} s_j - s_n$. Clearly, the equality holds. Else, we have $B = \operatorname{diag}(s_1, \ldots, s_n) \in \mathcal{S}(s_1, \ldots, s_n)$ attaining $\sum_{j=1}^{n-1} s_j - s_n$. The result follows. □

Recall that for two real vectors $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n)$, we say that $\mathbf{x} \prec_w \mathbf{y}$ is the sum of the k largest entries of $\mathbf{x}$ is not larger than that of $\mathbf{y}$ for $k = 1, \ldots, n$.

Theorem 3.6.2 *Let $d_1, \ldots, d_n$ be complex numbers such that $|d_1| \geq \cdots \geq |d_n|$. Then there is $A \in M_n$ with diagonal entries $d_1, \ldots, d_n$ and singular values $s_1 \geq \cdots \geq s_n$ if and only if*

$$(|d_1|, \ldots, |d_n|) \prec_w (s_1, \ldots, s_n) \qquad \textit{and} \qquad \sum_{j=1}^{n-1} |d_j| - |d_n| \leq \sum_{j=1}^{n-1} s_j - s_n.$$

Proof. The necessity follows from the previous theorem. We prove the converse by induction on $n \geq 2$. We will focus on the construction of A with singular values $s_1, \ldots, s_n$, and diagonal entries $d_1, \ldots, d_{n-1}, d_n$ with $d_1, \ldots, d_n \geq 0$.

Suppose $n = 2$. We have $d_1 + d_2 \leq s_1 + s_2, d_1 - d_2 \leq s_1 - s_2$. Let $A = \begin{pmatrix} d_1 & a \\ -b & d_2 \end{pmatrix}$ such that $a, b \geq 0$ satisfies $ab = s_1 s_2 - d_1 d_2$ and $a^2 + b^2 = s_1^2 + s_2^2 - d_1^2 - d_2^2$. Such a, b exist because

$$2(s_1 s_2 - d_1 d_2) = 2ab \leq a^2 + b^2 = s_1^2 + s_2^2 - d_1^2 - d_2^2.$$

Suppose the result holds for matrices of sizes up to $n-1 \geq 2$. Consider $(d_1, \ldots, d_n)$ and $(s_1, \ldots, s_n)$ that satisfy the inequalities. Let k be the largest integer k such that $s_k \geq d_1$.

If $k \leq n-2$, there is $B = \begin{pmatrix} d_1 & * \\ * & \hat{s} \end{pmatrix}$ with singular values s_k, s_{k+1}, where $\hat{s} = s_k + s_{k+1} - d_1$. One can check that $(d_2, \ldots, d_n)$ and $(s_1, \ldots, s_{k-1}, \hat{s}, s_{k+2}, \ldots, s_n)$ satisfy the inequalities for the $n-1$ case so that there are unitary $U, V \in M_{n-1}$ such that UDV^* has diagonal entries $d_2, \ldots, d_n$, where $D = \operatorname{diag}(\hat{s}, s_1, \ldots, s_{k-1}, s_{k+2}, \ldots, s_n)$. Thus,

$$A = ([1] \oplus U)(B \oplus \operatorname{diag}(s_1, \ldots, s_{k-1}, s_{k+2}, \ldots, s_n)([1] \oplus V^*)$$

has diagonal entries $d_1, \ldots, d_n$ and singular values $s_1, \ldots, s_n$.

Now suppose $k \geq n-1$, let

$$\hat{s} = \max \left\{ 0, d_n + s_n - s_{n-1}, \sum_{j=1}^{n-1} d_j - \sum_{j=1}^{n-2} s_j \right\} \leq \min \left\{ s_{n-1}, s_{n-1} + s_n - d_n, \sum_{j=1}^{n-2} (s_j - d_j) + d_{n-1} \right\}.$$

It follows that

$$(d_n, \hat{s}) \prec_w (s_{n-1}, s_n), \qquad |d_n - \hat{s}| \leq s_{n-1} - s_n,$$

$$(d_1, \ldots, d_{n-1}) \prec_w (s_1, \ldots, s_{n-2}, \hat{s}) \quad \text{and} \quad \sum_{j=1}^{n-2} d_j - d_{n-1} \leq \sum_{j=1}^{n-2} s_j - \hat{s}.$$

So, there is $C \in M_2$ with singular values s_{n-1}, s_n and diagonal elements $\hat{s}, d_n$. Moreover, there are unitary matrix $X, Y \in M_{n-1}$ such that $X\operatorname{diag}(s_1, \ldots, s_{n-2}, \hat{s})Y^*$ has diagonal entries $d_1, \ldots, d_{n-1}$. Thus,

$$A = (X \oplus [1])(\operatorname{diag}(s_1, \ldots, s_{n-2}) \oplus C)(Y^* \oplus [1])$$

will have the desired diagonal entries and singular values. □

3.7 Final remarks

The study of matrix inequalities has a long history and is still under active research. One of the most interesting question raised in 1960's and was finally solved in 2000's is the following.

Problem Determine the necessary and sufficient conditions for three set of real numbers $a_1 \geq \cdots \geq a_n, b_1 \geq \cdots \geq b_n, c_1 \geq \cdots \geq c_n$ for the existence of three (real symmetric) Hermitian matrices A, B and $C = A + B$ with these numbers as their eigenvalues, respectively.

It was proved that the conditions can be described in terms of the equality $\sum_{j=1}^n (a_j + b_j) = \sum_{j=1}^n c_j$ and a family of inequalities of the form

$$\sum_{j=1}^k (a_{u_j} + b_{v_j}) \geq \sum_{j=1}^k c_{w_j}$$

for certain subsequences $(u_1, \dots, u_k), (v_1, \dots, v_k), (w_1, \dots, w_k)$ of $(1, \dots, n)$.

There are different ways to specify the subsequences. A. Horn has the following recursive way to define the sequences.

1. If $k = 1$, then $w_1 = u_1 + v_1 - 1$. That is, we have $a_u + b_v \geq c_{u+v-1}$.
2. Suppose $k < n$ and all the subsequences of length up to $k - 1$ are specified. Consider subsequences $(u_1, \dots, u_k), (v_1, \dots, v_k), (v_1, \dots, v_k)$ satisfying $\sum_{j=1}^k (u_j + v_j) = \sum_{j=1}^k w_j + k(k+1)/2$, and for any lenth ℓ specified subsequences $(\alpha_1, \dots, \alpha_\ell), (\beta_1, \dots, \beta_\ell), (\gamma_1, \dots, \gamma_\ell)$ of $(1, \dots, n)$ with $\ell < k$,
$$\sum_{j=1}^{\ell} (u_{\alpha_j} + v_{\beta_j}) \geq \sum_{j=1} w_{\gamma_j}.$$

Consequently, the subsequences $(u_1, \dots, u_k), (v_1, \dots, v_k), (w_1, \dots, w_k)$ of $(1, \dots, n)$ is a Horn's sequence triples of length k if and only if there are Hermitian matrices $U, V, W = U + V$ with eigenvalues

$$u_1 - 1 \leq u_2 - 2 \leq \cdots \leq u_k - k, v_1 - 1 \leq v_2 - 2 \leq \cdots \leq v_k - k, w_1 - 1 \leq w_2 - 2 \leq \cdots \leq w_k - k,$$

respectively. This is known as the saturation conjecture/theorem.

Special cases of the above inequalities includes the following inequalities of Thompson, which reduces to the Weyl's inequalities when $k = 1$.

Theorem 3.7.1 *Suppose $A, B, C = A + B \in M_n$ are Hermitian matrices with eigenvalues $a_1 \geq \cdots \geq a_n, b_1 \geq \cdots b_n$ and $c_1 \geq \cdots \geq c_n$, respectively. For any subequences $(u_1, \dots, u_k), (v_1, \dots, v_k)$, of $(1, \dots, n)$, if $(w_1, \dots, w_k)$ is such that $w_j = u_j + v_j - j \leq n$ for all $j = 1, \dots, k$, then*

$$\sum_{j=1}^k (a_{u_j} + b_{v_j}) \geq \sum_{j=1}^k c_{w_j}.$$

Proof. We prove the result by induction on n. Suppose $n = 2$. If $k = n$ so that $(u_1, u_2) = (v_1, v_2) = (1, 2)$, then the equality holds. If $k = 1$, then $a_i + b_j \geq c_{i+j-1}$ for any $i + j \leq 3$ by the Lidskii inequality.

Now, suppose the result holds for all matrices of size $n - 1$. If $k = n$ so that $(u_1, \ldots, u_n) = (v_1, \ldots, v_n)$, then the equality holds. Suppose $k < n$. Let p be the largest integer such that $u_j = j$ for $j = 1, \ldots, p$, and let q be the largest integer such that $v_j = j$ for $j = 1, \ldots, q$. We may assume that $q \leq p < n$. Else, interchange the roles of A and B.

Let $\{y_1, \ldots, y_n\}$ be an orthonormal set of eigenvectors of B and $\{z_1, \ldots, z_n\}$ be an orthonormal set of eigenvectors of C so that

$$By_j = a_j y_j, \qquad Cz_j = z_j, \qquad j = 1, \ldots, n.$$

Suppose $Z \in M_{n,n-1}$ has orthonormal columns such that the column space of Z contains $z_1, \ldots, z_q, y_{q+2}, \ldots, y_n$. Let $\tilde{A} = Z^*AZ, \tilde{B} = Z^*BZ, \tilde{C} = Z^*CZ$ have eigenvalues $\hat{a}_1 \geq \cdots \geq \hat{a}_{n-1}, \hat{b}_1 \geq \cdots \geq \hat{b}_{n-1}$, and $\hat{c}_1 \geq \cdots \geq \hat{c}_{n-1}$, respectively. By induction assumption,

$$\sum_{j=1}^{q} \hat{c}_{u_j+v_j-j} + \sum_{j=q+1}^{k} \hat{c}_{u_j+(v_j-1)-j} \leq \sum_{j=1}^{k} \hat{a}_{u_j} + \sum_{j=1}^{k} \hat{b}_j + \sum_{j=q+1}^{k} b_{u_j+(v_j-1)-j}.$$

Because $u_j + v_j - j = j$ for $j = 1, \ldots, q$, and the column space of Z contains $z_1, \ldots, z_q$, we see that $\hat{c}_j = c_j$ for $j = 1, \ldots, q$. For $j = q + 1, \ldots, k$, we have $c_{u_j+v_j-j} \leq \hat{c}_{u_j+v_j-j-1}$, and hence

$$\sum_{j=1}^{q} c_{u_j+v_j-1} + \sum_{j=q+1}^{k} c_{u_j+v_j-j} \leq \sum_{j=1}^{q} c_{u_j+v_j-1} + \sum_{j=q+1}^{k} \hat{c}_{u_j+v_j-j-1}.$$

Because $\hat{b}_j \leq b_j$ for $j = 1, \ldots, q$, and $\hat{b}_{u_j+v_j-j-1} = b_{u_j+v_j-j}$ for $j = q + 1, \ldots, k$ as the column spaces contains $y_{q+1}, \ldots, y_n$, we have

$$\sum_{j=1}^{q} \hat{b}_j + \sum_{j=q+1}^{k} b_{u_j+(v_j-1)-j} \leq \sum_{j=1}^{k} b_{u_j+v_j-j}.$$

The result follows. □

Applying the result to $-A - B = -C$, we see that for any subsequences $(u_1, \ldots, u_k), (v_1, \ldots, v_k)$ and $(w_1, \ldots, w_k)$ with $w_j = u_i + v_j - j$ such that $u_k + v_k - k \leq n$, we have

$$\sum_{j=1}^{k} (a_{n-u_j+1} + b_{n-v_j+1}) \leq \sum_{j=1}^{k} (c_{n-w_j+1}).$$

Additional results and exercises

1. Suppose $n = 3$. List all the Horn sequences $(u_1, u_2), (v_1, v_2), (w_1, w_2)$ of length 2, and list all the Thompson standard sequences $(u_1, u_2), (v_1, v_2)$ and $(w_1, w_2) = (u_1 + v_1 - 1, u_2 + v_2 - 2)$.

2. Suppose $A, B, C = A + B \in M_n$ are Hermitian matrices have eigenvalues $a_1 \geq \cdots \geq a_n, b_1 \geq \cdots \geq b_n$ and $c_1 \geq \cdots \geq c_n$, respectively. Show that if $C = (c_{ij})$ then $\sum_{j=1}^k c_{jj} \leq \sum_{j=1}^k (a_j + b_j)$; the equality holds if and only if $A = A_{11} \oplus A_{22}, B = B_{11} \oplus B_{22}$ with $A_{11}, B_{11} \in M_k$ such that A_{11} and B_{11} have eigenvalues $a_1 \geq cdots \geq a_k, b_1 \geq \cdots \geq b_k$, respectively.

3. (Weyl's inequalities.) Suppose $A, B, C = A + B \in M_n$ are Hermitian matrices. For any $u, v \in \{1, \ldots, n\}$ with $u + v - 1 \leq n$, show that $\lambda_u(A) + \lambda_v(B) \geq \lambda_{u+v-1}(A + B)$.

   Hint: By induction on $n \geq 2$. Check the case for $n = 2$. Assume the result hold for matrices of size $n - 1$. Assume $v \leq u$. Let $\{z_1, \ldots, z_n\}$ and $\{y_1, \ldots, y_n\}$ be orthonormal sets such that $By_j = b_j y_j$ and $Cz_j = c_j z_j$ for $j = 1, \ldots, n$. If $Z \in M_{n,n-1}$ with orthonormal columns such that the column space of Z contains $y_1, \ldots, y_u$ and $z_{q+2}, \ldots, z_n$. Argue that

$$c_{u+v-1} = \lambda_{u+v-2}(Z^*CZ) \leq \lambda_u(Z^*AZ) + \lambda_{v-1}(Z^*BZ) \leq a_u + b_v.$$

4. Suppose $C = A + iB$ such that A and B has eigenvalues $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ such that $|a_1| \geq \cdots \geq |a_n|$ and $|b_1| \geq \cdots \geq |b_n|$. Show that if C has singular values $s_1, \ldots, s_n$, then

$$(a_1^2 + b_n^2, \ldots, a_n^2 + b_1^2) \prec (s_1^2, \ldots, s_n^2) \quad \text{and} \quad (s_1^2 + s_n^2, \ldots, s_n^2 + s_1^2)/2 \prec (a_1^2 + b_1^2, \ldots, a_n^2 + b_n^2).$$

   Hint: $2(A^2 + B^2) = CC^* + C^*C$.

5. Suppose $c_1 \geq a_1 \geq c_2 \geq a_2 \geq \cdots \geq a_{n-1} \geq c_n \geq a_n$ are $2n$ real numbers. Show that there is a nonnegative real vector $v \in \mathbb{R}^n$ such that $D + vv^t$ has eigenvalues $c_1 \geq \cdots \geq c_n$ for $D = \operatorname{diag}(a_1, \ldots, a_n)$.

   Hint: Replace c_j by $c_j + \gamma$ and $a_j + \gamma$ for $j = 1, \ldots, n$, for a sufficiently large $\gamma > 0$, and assume that $c_n \geq a_n > 0$. By interlacing inequalities, there is $\tilde{C} = \begin{pmatrix} D & y \\ y^t & a \end{pmatrix}$. Show that $C = D + vv^t$ has eigenvalues $c_1 \geq \cdots \geq c_n$.

6. Suppose $A = \begin{pmatrix} \tilde{A} & * \\ 0 & * \end{pmatrix}$. Show that

$$s_1(A) \geq s_1(\tilde{A}) \geq s_2(A) \geq s_2(\tilde{A}) \geq \cdots \geq s_{n-1}(\tilde{A}) \geq s_n(A).$$

7. (Extra credit) Suppose $A, B \in M_n$. For any subsequences $(u_1, \ldots, u_k), (v_1, \ldots, v_k)$ and $(w_1, \ldots, w_k)$ of $(1, \ldots, n)$ such that $w_j = u_j + v_j - j$ for $j = 1, \ldots, k$, and $u_k + v_k - k \leq n$, we have

$$\prod_{j=1}^k s_{u_j}(A) s_{v_j}(B) \geq \prod_{j=1}^k s_{w_j}(AB).$$

Hint: By induction on n. Check the case for $n = 2$. Assume that the result holds for matrices of size $n-1$. If $k = n$, the equality holds. Suppose $k < n$. Let p be the largest integer such that $u_j = j$ for all $j = 1, \ldots, p$, and q be the largest integer such that $v_j = j$ for all $j = 1, \ldots, q$. We may assume that $q \leq p$. Let $C = AB$, $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ be orthonormal sets such that

$$B^*Bu_j = s_j(B)^2 u_j \quad \text{and} \quad C^*Cv_j = s_j(C)^2 v_j.$$

Suppose U, V are unitary such that the first $n-1$ columns span a subspace containing $v_1, \ldots, v_1, u_{q+2}, \ldots, u_n$, and $V^*BU = \begin{pmatrix} \tilde{B} & * \\ 0 & * \end{pmatrix}$ with $\tilde{B} \in M_{n-1}$. Let W be unitary such that $W^*BV = \begin{pmatrix} \tilde{A} & * \\ 0 & * \end{pmatrix}$. Then $W^*ABV = \begin{pmatrix} \tilde{A}\tilde{B} & * \\ 0 & * \end{pmatrix}$. Apply induction assumption on $\tilde{A}\tilde{B}$ to finish the proof.

4 Norms

In many applications of matrix theory such as approximation theory, numerical analysis, quantum mechanics, one has to determine the "size" of a matrix, how near is one matrix to another, or how close is a matrix to a special class of matrices. We need concept of the norm (size) of a matrix. There are different ways to define the norm of a matrix, and the different definitions are useful in different applications.

4.1 Basic definitions and examples

Definition 4.1.1 *Let V be a linear space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A function $\nu : V \to [0, \infty)$ if*

(a) $\nu(v) \geq 0$ *for all $v \in V$; the equality holds if and only if $v = 0$.*

(b) $\nu(cv) = |c|\nu(v)$ *for any $c \in \mathbb{F}$ and $v \in V$.*

(c) $\nu(u + v) \leq \nu(u) + \nu(v)$ *for all $u, v \in V$.*

Example 4.1.2 *Let $V = \mathbb{F}^n$. For $v = (v_1, \ldots, v_n)^t \in \mathbb{F}^n$, let*

$$\ell_\infty(v) = \max\{|v_j| : 1, \ldots, n\} \quad \text{and} \quad \ell_p(v) = (\sum_{j=1}^{n} |v_j|^p)^{1/p} \quad \text{for } p \geq 1$$

be the ℓ_∞ nrom and the ℓ_p norm.

Note that $\ell_2(v) = (\sum_{j=1}^n |v_j|^2)^{1/2}$ is the inner product norm.

For every $p \in [1, \infty]$, it is easy to verify (a) and (b). For $p = 1, \infty$, it is easy to verify the triangular inequality. For $p > 1$, the verification of $\ell_p(u+v) \leq \ell_p(u) + \ell_p(v)$ is not so easy. We may change all the entries of u and v to their absolute values, and focus on vectors with nonnegative entries. to prove that the ℓ_p norm satisfies the triangle inequality $1 < p$, we establish the following.

Lemma 4.1.3 (Hölder's inequality) *Let $p, q > 1$ be such that $1/p + 1/q = 1$. For $u = (u_1, \ldots, u_n)^t$ and $v = (v_1, \ldots, v_n)^t$ with positive entries,*

$$\sum_{j=1} u_j v_j \leq \ell_p(u)\ell_q(v).$$

The equality holds if and only if $(u_1^p, \ldots, u_n^p)^t$ and $(v_1^q, \ldots, v_n^q)^t$ are linearly dependent.

Proof. Replace (u, v) by $(u/\ell_p(u), v/\ell_q(v))$. We need to show that $u^t v \leq 1$. Note that for two positive numbers a, b, we have

$$ab = \exp(\ln a + \ln b) = \exp((1/p)\ln(a^p) + (1/q)\ln(b^q))$$

$$\leq (1/p)\exp(\ln(a^p)) + (1/q)\exp(\ln(b^q)) = a^p/p + b^q/q,$$

where the equality holds if and only if $a^p = b^q$. Thus, we have $u_k v_k \le u_k^p/p + v_k^q/q$, and

$$\sum_{j=1}^{n} u_j v_j \le \ell_p(u)/p + \ell_q(v)/q = 1,$$

where the equality holds if and only if $u_j^p = v_j^q$ for all $j = 1, \ldots, n$. □

Corollary 4.1.4 (Minkowski inequality) *Suppose* $p \in [1, \infty]$. *We have* $\ell_p(u+v) \le \ell_p(u) + \ell_p(v)$.

Proof. The cases for $p = 1, \infty$ can be readily checked. Suppose $p > 1$. By the Hölder inequality, if $1 - 1/p = 1/q$, then

$$\begin{aligned}\sum_{j=1}^{n}(u_j+v_j)^p &= \sum_{j=1}^{n} u_j(u_j+v_j)^{p-1} + v_j(u_j+v_j)^{p-1} \\ &\le \ell_p(u)(\sum_{j=1}^{n}(u_j+v_j)^p)^{1/q} + \ell_p(v)(\sum_{j=1}^{n}(u_j+v_j)^p)^{1/q} \qquad \text{as } (p-1)q = p \\ &= (\ell_p(u)+\ell_p(v))\left(\sum_{j=1}^{n}(u_j+v_j)^p\right)^{1/q}.\end{aligned}$$

Dividing both sides by $\left(\sum_{j=1}^{n}(u_j+v_j)^p\right)^{1/q}$, we get the conclusion. □

Next, we consider examples on matrices.

Example 4.1.5 Consider $V = M_{m,n}$. Using the inner product $\langle A, B\rangle = \mathrm{tr}\,(AB^*)$ on $M_{m,n}$, we have the inner product norm (a.k.a. Frobenius norm)

$$\|A\| = (\mathrm{tr}\, AA^*)^{1/2} = (\sum_{i,j}|a_{ij}|^2)^{1/2} = \sum_{j=1}^{m} s_j(A)^2.$$

One can define the $\ell_p(A) = (\sum_{i,j}|a_{ij}|^p)^{1/p}$, and define the Schatten p-norm by

$$S_p(A) = \ell_p(s(A)) = (\sum_{j=1}^{m} s_j(A)^p)^{1/p}.$$

The Schatten ∞-norm reduces to $s_1(A)$, which is also known as the spectral norm or operator norm defined by

$$\|Ax\| = \max\{\ell_2(Ax) : x \in \mathbb{C}^n, \ell_2(x) \le 1\}.$$

When $m = n$, the Schatten 1-norm of A is just the sum of the singular values of A, and is also known as the trace norm.

One can also define the Ky Fan k-norm by $F_k(A) = \sum_{j=1}^{k} s_j(A)$ for $k = 1, \ldots, m$.

Assertion *The Ky Fan k-norms and the Schatten p-norms satisfy the triangle inequalities.*

Proof. To prove the triangle inequality for the Ky Fan k-norm, note that if $C = A + B$, then $\begin{pmatrix} 0 & C \\ C^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} + \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}$. By the Lidskii inequalities $\sum_{j=1}^k s_j(C) \le \sum_{j=1}^k (s_j(A) + s_j(B))$. So, we have proved $s(C) \prec_w s(A) + s(B)$. By the homework exercise, we have

$$s_p(C) = \ell_p(s(C)) \le \ell_p(s(A) + s(B) \le \ell_p(s(A)) + \ell_p(s(B)) = S_p(A) + s_p(B).$$

□

For $A \in M_n$, one can define the numerical range and numerical radius of A by

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n,\ x^*x = 1\} \quad \text{and} \quad w(A) = \max\{|x^*Ax| : x \in \mathbb{C}^n,\ x^*x = 1\},$$

respectively. The spectral radius of $A \in M_n$ as

$$r(A) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}.$$

Example 4.1.6 *If* $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, *then*

$$\begin{aligned} W(A) &= \{(\bar{x}_1, \bar{x}_2)A(x_1, x_2)^t : |x_1|^2 + |x_2|^2 = 1\} = \{2\bar{x}_1 x_2 : |x_1|^2 + |x_2|^2 = 1\} \\ &= \{2\cos\theta \sin\theta e^{it} : \theta \in [0, \pi/2], t \in [0, 2\pi)\} = \{\mu \in \mathbb{C} : |\mu| \le 1\} \end{aligned}$$

Note that the numerical radius is a norm on M_n (homework), but the spectral radius is not.

Theorem 4.1.7 *Let $A \in M_n$. Then $W(A)$ is a compact convex set containing all the eigenvalues of A, and*

$$r(A) \le w(A) \le s_1(A) \le 2w(A).$$

Proof. Let $x, y \in \mathbb{C}^n$ be unit vectors, and $\alpha = x^*Ax, \beta = y^*Ay \in W(A)$. We need to show that the line segment joining α and β lies in $W(A)$. We assume $\alpha \ne \beta$ to avoid trivial consideration.

Note that $W(\xi A + \mu I) = \xi W(A) + \mu = \{\xi x^*Ax + \mu : x \in \mathbb{C}^n, x^*x = 1\}$. We may replace A by $B = (A - \alpha I)/(\beta - \alpha)$, and show that the line joining $x^*Bx = 0$ and $y^*By = 1$ lies in $W(B)$. We may further assume that $x^*By + y^*Bx \in \mathbb{R}$. Else, replace y by $e^{ir}y$ for a suitable $r \in [0, 2\pi)$.

Now, let $z(s) = [(1-s)x + sy]/\|(1-s)x + sy\|$ so that

$$z(s)^*Bz(s) = \frac{(1-s)^2 x^*Bx + s(1-s)(x^*By + y^*Bx) + s^2 y^*By}{\|(1-s)x + sy\|^s} \in W(B), \qquad s \in [0, 1],$$

has real values vary from 0 to 1 continuously as s varies in $[0, 1]$. So, $[0, 1] \subseteq W(B)$.

The set $W(A)$ is compact means that it is bounded and contains all the boundary points. It follows from the fact that $W(A)$ is the image of the set of unit vectors in $\mathbb{C}^n$ under the continuous function $x \mapsto x^*Ax$.

Now, if λ is an eigenvalue of A, let x be a corresponding unit eigenvector of λ, then $x^*Ax = \lambda \in W(A)$. So, $r(A) \le w(A)$. Also, we have

$$w(A) = \max\{|x^*Ax| : x \in \mathbb{C}^n, x^*x = 1\} \le \max\{|x^*Ay| : x, y \in \mathbb{C}^n, x^*x = y^*y = 1\} \le s_1(A).$$

Finally, if $A = H + iG$ with $H = H^*, G = G^*$, then there are unit vectors $x, y \in \mathbb{C}^n$ such that

$$s_1(A) \le s_1(H + iG) \le s_1(H) + s_1(G) = |x^*Hx| + |y^*Gy| \le |x^*Ax| + |y^*Ay| \le 2w(A).$$

□

Definition 4.1.8 *A norm $\|\cdot\|$ on M_n is a matrix/algebra norm if*

$$\|AB\| \le \|A\|\|B\| \qquad \text{for all } A, B \in M_n.$$

Suppose ν is a norm on $\mathbb{F}^n$. Then the operator norm induced by ν is defined by

$$\|A\|_\nu = \max\{\nu(Ax) : x \in \mathbb{C}^n, \nu(x) \le 1\}.$$

Note that every induced norm is a matrix norm. The Schatten p-norms, the Ky Fan k-norms, are matrix norms, but the numerical radius is not.

Example 4.1.9 *The operator norm induced by the ℓ_1-norm on $\mathbb{F}^n$ is the column sum norm defined by*

$$\|A\|_{\ell_1} = \max\{\sum_{j=1}^n : |a_{j\ell}| : \ell = 1, \dots, n\}.$$

The operator norm induced by the ℓ_∞-norm on $\mathbb{F}^n$ is the row sum norm defined by

$$\|A\|_{\ell_\infty} = \max\{\sum_{j=1}^n : |a_{\ell j}| : \ell = 1, \dots, n\}.$$

Theorem 4.1.10 *Let $A \in M_n$. Then $\lim_{k\to\infty} A^k = 0$ if and only if $r(A) < 1$.*

Proof. Let $A = S(J_1 \oplus \cdots \oplus J_k)S^{-1}$, where $J_1, \dots, J_k$ are Jordan blocks. Assume $r(A) < 1$. We will show that $A^\ell \to 0$ as $\ell \to \infty$. It suffices to show that $J_i^\ell \to 0$ as $\ell \to \infty$ for each $i = 1, \dots, k$.

Note that if μ satisfies $|\mu| < 1$ and $N_m = E_{12} + \cdots E_{m-1,m} \in M_m$, then for $\ell > m$,

$$(\mu I_m + N_m)^\ell = \sum_{j=0}^{m-1} \binom{\ell}{p} \mu^{\ell-p} N^p \to 0 \qquad \text{as} \quad \ell \to \infty$$

as $\lim_{\ell\to\infty} \binom{\ell}{p}\mu^{\ell-p} = 0$. Conversely, if $Ax = \mu x$ for some $|\mu| \ge 1$ and unit vector $x \in \mathbb{C}^n$, then $A^k x = \mu^k x$ so that $A^k \not\to 0$ as $k \to \infty$. □

Theorem 4.1.11 *Let $\|\cdot\|$ be a matrix norm on M_n. Then*

$$\lim_{k\to\infty} \|A^k\|^{1/k} = r(A).$$

Proof. Suppose μ is an eigenvalue of A such that $|\mu| = r(A)$. Let x be a unit vector such that $Ax = \mu x$. Then $|\mu|\|[x\cdots x]\| = \|A[x\cdots x]\| \le \|A\|\|[x\cdots x]\|$. So, $|\mu| \le \|A\|$ and $\|\mu^k| \le \|A^k\|$.

Now, for any $\varepsilon > 0$, let $A_\varepsilon = A/(r(A)+\varepsilon)$. Then $\lim_{k\to\infty} A_\varepsilon^k = 0$. So, for sufficiently large $k \in \mathbb{N}$ we have $\|A^k/(rA+\varepsilon)^k\| < 1$. Hence, for any $\varepsilon > 0$, if k is sufficiently large, then

$$r(A) \le \|A^k\|^{1/k} \le r(A) + \varepsilon.$$

The result follows. □

Remark In the proof, we use the fact that the function $x \mapsto \|x\|$ is continuous. To see this, for any $\varepsilon > 0$, we can let $\delta = \varepsilon$, then $\|x - y\| < \delta$, we have $|\|x\| - \|y\|| \le \|x - y\| = \delta = \varepsilon$.

Corollary 4.1.12 *Suppose $\|\cdot\|$ is a matrix norm on M_n such that $\|A\| \ge r(A)$ for all $A \in M_n$. If $\|A\| < 1$, then $A^k \to 0$ as $k \to \infty$.*

4.2 Geometric and analytic properties of norms

Let ν be a norm on a linear space V. Then

$$\mathcal{B}_\nu = \{x \in V : \nu(x) \le 1\}$$

is the unit ball of the norm ν.

Theorem 4.2.1 *Let ν be a norm on a nonzero linear space V. Then $\mathcal{B}_v$ satisfies the following.*

(a) *The zero vector 0 is an interior point.*

(b) *For any $\mu \in \mathbb{F}$ with $|\mu| = 1$,*

$$\mathcal{B}_v = \mu\mathcal{B}_\nu = \{\mu x : x \in \mathcal{B}_\nu\}.$$

(c) *The set $\mathcal{B}_\nu$ is convex. That is if $x, y \in \mathcal{B}_\nu$, then $tx + (1-t)y \in \mathcal{B}_\nu$.*

Conversely, if V is finite dimensional linear space over $\mathbb{F}$ and $\mathcal{B}$ is a set satisfying (a) — (c)*, then we can define a norm $\|\cdot\|$ on V by $\|x\| = 0$, and for any nonzero $x \in V$,*

$$\|x\| = \sup\{t > 0 : x/t \in \mathcal{B}\} = \max\{t > 0 : x/t \in \mathcal{B}\}.$$

Theorem 4.2.2 *Suppose ν_j for $j \in J$ is a family of norm on a linear space V so that 0 is an interior point of $\cap\mathcal{B}_{\nu_j}$. Then $\cap\mathcal{B}_{\nu_j}$ is the unit norm ball of ν defined by*

$$\nu(x) = \sup\{\nu_j(x) : j \in J\}.$$

4.3 Inner product norm and the dual norm

Recall that for a linear space V, a scalar function on $V \times V$ is an inner product denoted by $\langle x, y\rangle \in \mathbb{F}$ if it satisfies

(a) $\langle x, x\rangle \geq 0$, where the equality holds if and only if $x = 0$,

(b) $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$,

(c) $\langle x, z\rangle = \overline{\langle z, x\rangle}$,

for any $a, b \in \mathbb{F}, x, y, z \in V$.

Theorem 4.3.1 *Suppose V is an inner product space. Then for any $x, y \in V$,*

$$\|x\| = \langle x, x\rangle^{1/2} \qquad x \in V$$

is a norm satisfying the Cauchy inequality

$$|\langle x, y\rangle| \leq \|x\|\|y\|$$

and the parallelogram identity

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Theorem 4.3.2 *Suppose $\|\cdot\|$ is a norm on a linear space V satisfying the parallelogram identity. Then one can define an inner product by $\langle x, y\rangle = a + ib$ with*

$$2a = \|x + y\|^2 - \|x\|^2 - \|y\|^2 \quad \text{and} \quad 2b = \|x + iy\|^2 - \|x\|^2 - \|y\|^2$$

such that $\|z\| = \langle z, z\rangle^{1/2}$ for all $z \in V$.

Remark 4.3.3 *Suppose V is an inner product space, and ν is a norm on V. One can define the dual norm on V by*

$$\nu^D(x) = \sup\{|\langle x, y\rangle| : \nu(y) \leq 1\}.$$

We have $(\nu^D)^D = \nu$.

Example 4.3.4 *The dual norm of the ℓ_p norm on $\mathbb{F}^n$ is the ℓ_p norm with $1/p + 1/q = 1$.*

The dual norm of the Schatten p norm on $M_{m,n}$ is the Schatten q norm on $M_{m,n}$ with $1/p+1/q = 1$.

The dual norm of the Ky Fan k-norm on $M_{m,n}$ with $m \geq n$ is $F_k^d(A) = \max\{\sum_{j=1}^n s_j(A), s_1(A)\}$

4.4 Symmetric norms and unitarily invariant norms

A norm on $\mathbb{F}^n$ is a symmetric norm if $\|x\| = \|Px\|$ for all permutation matrix P or diagonal unitary (orthogonal) matrix P.

A norm on $M_{m,n}(\mathbb{F})$ is a unitarily invariant norm (UI norm) if $\|UAV\| = \|A\|$ for any unitary $U \in M_m, V \in M_n$, and any $A \in M_{m,n}$.

Theorem 4.4.1 *Suppose $m \geq n$. Every UI norm $\|\cdot\|$ on $M_{m,n}$ corresponds to a symmetric norm ν on $\mathbb{R}^n$ such that*

$$\|A\| = \nu(s(A)) \qquad \text{for all } A \in M_{m,n}.$$

Proof. Suppose $\|\cdot\|$ is a UI norm. Then $\|A\| = \|\sum_{j=1}^n s_j(A)E_{jj}\|$ for any $A \in M_{m,n}$. Define $\nu : \mathbb{F}^n \to \mathbb{R}$ by $\nu(x) = \|\sum_{j=1}^n |x_j|E_{jj}\|$ for $x = (x_1, \ldots, x_n)^t \in \mathbb{R}^n$. Then it is easy to verify that ν is a symmetric norm.

Conversely, if ν is a symmetric norm on $\mathbb{R}^n$, then define $\|\cdot\|$ by $\|A\| = \nu(s(A))$ for any $A \in M_{m,n}$. Then one can check that $\|A\|$ is a norm using the fact that $s(A+B) \prec s(A) + s(B)$ so that $\nu(s(A+B)) \leq \nu(s(A) + s(B))$. □

Denote by GP_n the set of matrices equal to the product of a permutation matrix and a diagonal unitary (orthogonal) matrices if $\mathbb{F} = \mathbb{C}$ (if $\mathbb{F} = \mathbb{R}$). Let $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$ with $c_1 \geq \cdots \geq c_n \geq 0$. Define the c-norm on $\mathbb{F}^n$ by

$$\nu_c(x) = \max\{c^t Px : P \in GP_n\}$$

and the c-spectral norm on $M_{m,n}(\mathbb{F})$ by

$$\|A\|_c = \nu_c(s(A)).$$

If $c = (\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0)$, we get the $\nu_k(x)$ and the Ky Fan k-norm $F_k(A)$.

Lemma 4.4.2 *Suppose ν on $\mathbb{R}^n$ is a symmetric norm. Then for any $x \in \mathbb{R}^n$,*

$$\nu(x) = \max\{\nu_c(x) : c = (c_1, \ldots, c_n), c_1 \geq \cdots \geq c_n, \nu^d(c) = 1\}.$$

Suppose $\|\cdot\|$ is a UI norm on $M_{m,n}(\mathbb{F})$. Then for any $A \in M_{m,n}$,

$$\|A\| = \max\{\|A\|_c : C = s(C) \text{ for some } C \in M_{m,n}, \|C\|^d = 1\}.$$

Theorem 4.4.3 *Let $x, y \in \mathbb{F}^n$. The following are equivalent.*

(a) $\nu_k(x) \leq \nu_k(y)$ *for all* $k = 1, \ldots, n$.

(b) $\nu_c(x) \leq \nu_c(y)$ *for all nonzero* $c = (c_1, \ldots, c_n)$ *with* $c_1 \geq \cdots \geq c_n \geq 0$.

(c) $\nu(x) \leq \nu(y)$ *for all symmetric norms* ν.

Proof. Suppose (a) holds. Then for any $c = (c_1, \ldots, c_n)$ with $c_1, \ldots, c_n$, if we set $d_n = c_n$ and $d_j = c_j - j + 1$ for $j = 1, \ldots, n-1$, then $\nu_c(z) = \sum_{j=1}^n d_j \nu_j(z)$. Thus,

$$\nu_c(x) = \sum_{j=1}^n d_j \nu_j(x) \leq \sum_{j=1}^n d_j \nu_j(y) = \sum_{j=1}^n c_j y_j = \nu_c(y).$$

Suppose (b) holds. Let ν be a symmetric norm. Then for any $c = (c_1, \ldots, c_n)$ with $c_1 \geq \cdots \geq c_n \geq 0$ with $\nu^d(c) = 1$, we have $\nu_c(x) \leq \nu_c(y)$. Thus, $\nu(x) = \nu(y)$.

The implication (b) $\Rightarrow$ (c) is clear. □

Theorem 4.4.4 *Let $A, B \in M_{m,n}(\mathbb{F}^n)$ with $m \geq n$. The following are equivalent.*

(a) *$F_k(A) \leq F_k(B)$ for all $k = 1, \ldots, n$.*

(b) *$\|A\|_c \leq \|B\|_c$ for all nonzero $c = (c_1, \ldots, c_n)$ with $c_1 \geq \cdots \geq c_n \geq 0$.*

(c) *$\|A\| \leq \|B\|$ for all UI norms $\|\cdot\|$.*

Proof. Similar to the last theorem. □

Theorem 4.4.5 *Let $\mathcal{R}_k \subseteq M_{m,n}$ be the set of matrices of rank at most k with $m \geq n > k$. Suppose $\|\cdot\|$ is a UI norm. If $A \in M_{m,n}$ such that $U^*AV = \sum_{j=1}^n s_j(A)E_{jj}$, then $A_k = U(\sum_{j=1}^k s_j(A)E_{jj})V^*$ satisfies*

$$\|A - A_k\| \leq \|A - X\| \qquad \text{for all } X \in \mathcal{R}_k.$$

Proof. Let $X \in \mathcal{R}_k$ and $C = A - X$. Then $s_j(X) = 0$ for $j > k$ so that

$$\sum_{j=1}^{\ell} s_{k+j}(A) = \sum_{j=1}^{\ell} (s_{k+j}(A) - s_{k+1}(X)) \leq \sum_{j=1}^{\ell} s_j(C), \qquad \text{for all } \ell = 1, \ldots, n-k.$$

So, $(s_{k+1}(A), \ldots, s_n(A), 0, \ldots, 0) \prec_w s(C)$ and $\|A - A_k\| \leq \|C\| = \|A - X\|$. □

Theorem 4.4.6 *Let $A \in M_n$ and $\|\cdot\|$ be a unitarily invariant norm.*

(a) *$\|A - (A + A^*)/2\| \leq \|A - H\|$ for any $H = H^* \in M_n$.*

(b) *$\|A - (A - A^*)/2\| \leq \|A - iG\|$ for any $G = G^* \in M_n$.*

Proof. (a) Let $H \in M_n$ be Hermitian, and let $A - H = \hat{H} + iG$. Suppose $Q \in M_n$ is unitary such that Q^*GQ is in diagonal form $g_1, \ldots, g_n$ such that $|g_1| \geq \cdots \geq |g_n|$. If $d_1, \ldots, d_n$ are the diagonal entries of $Q^*(H + iG)Q$, then

$$s(G) = (|g_1|, \ldots, |g_n|) \prec_w (|d_1|, \ldots, |d_n|) \prec_w (A - H).$$

Thus, $\|G\| = \|A - (A + A^*)/2\| \leq \|A - H\|$.

(b) Similar to (a). □

Theorem 4.4.7 *Suppose $A, B \in M_n$ have singular values $a_1 \geq \cdots \geq a_n$ and $b_1 \geq \cdots \geq b_n$. Then for any UI norm $\|\cdot\|$,*

$$\|\sum_{j=1}^{m}(a_j + b_{n-j+1})E_{jj}\| \leq \|A + B\| \leq \|\sum_{j=1}^{m}(a_j + b_j)E_{jj}\|.$$

Proof. By Lidskii inequalities, for all $k = 1, \ldots, n$,

$$\sum_{j=1}^{k}[\lambda_j(A+B) - \lambda_j(A)] \leq \sum_{j=1}^{k}\lambda_j(B) \quad \text{and} \quad \sum_{j=1}^{k}\lambda_{i_j}(A) - \lambda_{i_j}(-B) \leq \sum_{j=1}^{k}\lambda_j(A - (-B)).$$

We get the majorization result. □

Theorem 4.4.8 *Let $\|\cdot\|$ be a UI norm on M_n.*

(a) *If P is positive semidefinite, then $\|P - I\| \leq \|P - V\| \leq \|P + V\|$ for any unitary $V \in M_n$.*

(b) *If $A = UP$ such that P is positive semidefinite and U is unitary, then*

$$\|A - U\| \leq \|A - V\| \qquad \text{for any unitary } V \in M_n.$$

Proof. (a) Apply the previous theorem with $P = A$ and $B = I$.

(b) Use the fact that $\|A - V\| = \|UP - V\| = \|P - U^*V\| \geq \|P - I\| = \|UP - U\|$. □

4.5 Errors in computing inverse and solving linear equations

Theorem 4.5.1 *If $B \in M_n$ satisfies $r(B) < 1$, then $I - B$ is invertible and*

$$(I - B)^{-1} = \sum_{k=0}^{\infty} B^k.$$

Consequently, if $A \in M_n$ is invertible and E satisfies $r(A^{-1}E) < 1$, then $A + E$ is invertible and

$$A^{-1} - (A + E)^{-1} = \sum_{k=1}^{\infty}(A^{-1}E)^k A^{-1}.$$

Furthermore, if $\|\cdot\|$ is a matrix norm on M_n such that $\|A^{-1}E\| < 1$ and $\kappa(A) = \|A^{-1}\|\|A\|$, then

$$\frac{\|A^{-1} - (A+E)^{-1}\|}{\|A^{-1}\|} \leq \frac{\|A^{-1}E\|}{1 - \|A^{-1}E\|} \leq \frac{\kappa(A)}{1 - \kappa(A)(\|E\|/\|A\|)}\frac{\|E\|}{\|A\|}.$$

Proof. Use the identity $(I-A)(\sum_{j=1}^k A^j) = I - A^{k+1}$ and letting $k \to \infty$. □

The quantity $\kappa(A)$ is called the condition number of A with respect to the norm $\|\cdot\|$.

Important implication, the change of the inverse will affected by $\kappa(A)$. For example, if $\|A\| = s_1(A)$, and A is unitary, then

$$\frac{\|A^{-1} - (A+E)^{-1}\|}{\|A^{-1}\|} \le \frac{\|E\|}{\|A\| - \|E\|}.$$

So, the computation of A is very "stable".

We can apply the result to analysis the solution of $Ax = b$.

Corollary 4.5.2 *Let $A, E \in M_n$ and $x, b \in \mathbb{C}^n$ be such that $Ax = b$ and $(A+E)\hat{x} = b$. Suppose A and $(A+E)$ are invertible.*

$$x - \hat{x} = [A^{-1} - (A+E)^{-1}]b = [A^{-1} - (A+E)^{-1}]A^{-1}x.$$

Suppose $\|\cdot\|$ is a matrix norm on M_n such that $\|A^{-1}E\| < 1$, and if ν is a norm on $\mathbb{C}^n$ such that $\nu(Bz) \le \|B\|\nu(z)$ for all $B \in M_n$ and $z \in \mathbb{C}^n$. If $\kappa(A) = \|A^{-1}\|\|A\|$, then

$$\frac{\nu(x-\hat{x})}{\nu(x)} \le \frac{\|A^{-1}E\|}{1 - \|A^{-1}E\|} \le \frac{\kappa(A)}{1 - \kappa(A)(\|E\|/\|A\|)} \frac{\|E\|}{\|A\|}.$$

5 Additional topics

5.1 Location of eigenvalues

Theorem 5.1.1 (Gershgorin Theorem) *Let $A \in (a_{ij})$, and let*

$$G_j(A) = \{\mu \in \mathbb{C} : |\mu - a_{jj}| \le \sum_{i \ne j} |a_{ji}|\}.$$

Then the eigenvalues of A lies in $G(A) = \cup_{j=1}^n G_j(A)$. Furthermore, if $C = G_{i_1}(A) \cup \cdots \cup G_{i_j}(A)$ form a connected component of G, then C contains exactly k eigenvalues counting multipicities.

Proof. Suppose $Av = \lambda v$ with $v = (v_1, \dots, v_n)$. Then for $i = 1, \dots, n$,

$$\lambda v_i - a_{ii} v_i = \sum_{j \ne i} a_{ij} v_j.$$

Suppose v_i has the maximum size. Then

$$|\lambda - a_{ii}| = |\sum_{j \ne i} a_{ij} v_j / v_i| \le \sum_{j \ne i} |a_{ij}|.$$

To prove the last assertion. Let $A_t = D + t(A - D)$ with $D = \text{diag}\,(a_{11}, \dots, a_{nn})$. Then A_0 has eigenvalues $a_{11}, \dots, a_{nn}$, and the eigenvalues and Gershgorin disk will change continuously according to $t \in [0, 1]$ until we get $A_1 = A$.

One can apply the result to A^t to get Gershgorin disks of different sizes centered at $a_{11}, \dots, a_{nn}$. Also, one can apply the result to $S^{-1}AS$ for (simple) invertible S such that $G(S^{-1}AS)$ is small. In fact, if A is already in Jordan form, then for any $\varepsilon > 0$ there is S such that $S^{-1}AS$ has diagonal entries $\lambda_1, \dots, \lambda_n$ and $(i, i+1)$ entries equal 0 or ε for $i = 1, \dots, n-1$, and all other entries equal to 0. So, we have the following.

Theorem 5.1.2 *Let $A \in M_n$. Then*

$$\bigcap_{S \in M_n \text{ is invertible}} G(S^{-1}AS) = \{\lambda_1(A), \dots, \lambda_n(A)\}.$$

One may use the Gershgorin theorem to study the zeros of a (monic) polynomial, namely, one can apply the result to the companion matrix C_f of $f(x)$ to get some estimate of the location of the zeros. One can further apply similarity to C_f to get better estimate for the zeros of $f(x)$.

5.2 Eigenvalues and principal minors

Theorem 5.2.1 *Let $A \in M_n$ with eigenvalues $\lambda_1, \dots, \lambda_n$. Then*

$$\det(zI - A) = (z - \lambda_1) \cdots (z - \lambda_n) = z^n - a_1 z^{n-1} + a_2 z^{n-2} - \cdots + (-1)^n a_n,$$

where for $m = 1, \dots, n$,

$$a_m = S_m(\lambda_1, \dots, \lambda_n) = \sum_{1 \le j_1 < \cdots < j_m \le n} (\lambda_{j_1} + \cdots + \lambda_{j_m})$$

is the sum of all $m \times m$ principal minors of A.

Proof. For any subseteq $J \subseteq \{1, \dots, n\}$, let $A[J]$ be the principal submatrix of A with row and column indices in J. Consider the expansion $\det(zI - A)$. The coefficient of z^{n-j} comes from the sum of the leading coefficients of $(-1)^j \det(A[J]) \det(zI - A[\overline{J}])$ for all different j-element subsets J of $\{1, \dots, n\}$. The result follows. $\square$

5.3 Nonnegative Matrices

In this section, we consider positive (nonnegative) matrices A, i.e., the entries of A are positive (nonnegative) real numbers. Denote by $|A|$ the matrix obtained from A by changing its entries to their absolute values (norm). Similarly, we consider $|v|$ of a vector v.

Theorem 5.3.1 (Perron-Frobenius Theorem) *Suppose $A \in M_n$ is nonnegative such that A^k is positive for some positive integer k. Then the following holds.*

(a) *$r(A) > 0$ is an algebraically simple eigenvalue of A such that $r(A) > |\lambda|$ for all other eigenvalue λ of A.*

(b) *There is a unique positive vector x with $\ell_1(x) = 1$ such that $Ax = r(A)x$, and there is a unique positive vector y with $y^t x = 1$ and $y^t A = r(A)y^t$.*

(c) *Let x and y be the vectors in* (b). *Then $(r(A)^{-1}A)^m \to xy^t$ as $m \to \infty$.*

We first prove a lemma.

Lemma 5.3.2 *Suppose $A \in M_n$ is nonnegative with row sums $r_1, \ldots, r_n$.*

(a) *For any nonnegative matrix P, $r(A) \le r(A + P)$.*

(b) *If all the row sums are the same, then $r(A) = r_1$. In general,*

$$\min\{r_i : 1 \le i \le n\} \le r(A) \le \max\{r_i : 1 \le i \le n\}.$$

Proof. (a) If $B = A + P$, then for any positive integer k, $B^k - A^k$ is nonnegative so that $\|A^k\|_{\ell_\infty} \le \|B^k\|_{\ell_\infty}$. Hence,

$$r(A) = \lim_{k\to\infty} \|A^k\|_{\ell_\infty}^{1/k} \le \lim_{k\to\infty} \|B^k\|_{\ell_\infty}^{1/k} = r(B).$$

(b) Suppose all the row sums are the same. Let $e = (1, \ldots, 1)^t$. Then $Ae = r_1 e$ so that r_1 is an eigenvalue. By Gershgorin Theorem all eigenvalues lie in

$$\bigcup_{i=1}^n \left\{\mu \in \mathbb{C} : |\mu - a_{ii}| \le \sum_{j\ne i} a_{ij}\right\}.$$

Thus, all eigenvalues lie in the set $\{\mu \in \mathbb{C} : |\mu| \le r_1\}$. Hence, $r_1 = r(A)$.

In general, let P be a nonnegative matrix such that $B = A + P$ has all row sum equal to $\|A\|_{\ell_\infty}$. Then $r(A) \le r(B) = \|A\|_{\ell_\infty}$.

Similarly, let Q be a nonnegative matrix such that $\hat{B} = A - Q$ is nonnegative with all row sum equal to $r_\ell = \min\{r_i : 1 \le i \le n\}$. Then $r_\ell = r(\hat{B}) \le r(A)$. □

Proof of Theorem 5.3.1. Assume $B = A^k$ is positive. Then $r(B)$ is larger than the minimum row sum of B so that $0 < r(B) = r(A)^k$. Note that Bv is positive for any nonzero vector $v \ge 0$.

Assertion 1 *Let λ be an eigenvalue of B. Either $|\lambda| < r(B)$ or $\lambda = r(B)$ with an eigenvector x such that $x = e^{i\theta}|x|$ for some $\theta \in \mathbb{R}$.*

Proof. Let λ be an eigenvalue of B such that $|\lambda| = r(B)$, and x be an eigenvector. Then $r(B)|x| = |r(B)x| = |Bx| \le B|x|$. We claim that the equality holds. If it is not true, we can set $z = B|x|$ so that $y = (B - r(B))|x| = z - r(B)|x| \ne 0$ is nonnegative. Then

$$0 < By = Bz - r(B)B|x| = Bz - r(B)z.$$

So, $z = (z_1, \dots, z_n)^t$ has positive entries, and for $Z = \operatorname{diag}(z_1, \dots, z_n)$, we have

$$Z^{-1}(BZe - r(B)Ze) = Z^{-1}BZe - r(B)e = Z^{-1}By > 0.$$

If follows that $Z^{-1}BZ$ has minimum row sum $r(B)+\delta$, where $\delta = \ell_\infty(Z^{-1}By) > 0$. So, $r(Z^{-1}BZ) \ge r(B) + \delta$, which is a contradiction.

Now, $r(B)|x| = B|x|$ has positive entries, and $|Bx| = r(B)|x| = B|x|$. Thus, $x = e^{i\theta}|x|$, i.e., x is the eigenspace of $r(B)$ and $\lambda = r(B)$. The proof of Assertion 1 is complete.

Assertion 2 *The value $r(B)$ is a simple eigenvalue of B with a unique positive positive eigenvector x satisfying $e^tx = 1$ and a unique positive left eigenvector y such that $y^tx = 1$. Moreover, there is an invertible matrix $S \in M_n$ such that x is the first column of S and y^t is the first row of S^{-1} satisfying $S^{-1}BS = [r(B)] \oplus B_1$ with $r(B_1) < r(B)$.*

Proof. Suppose $Bu = r(B)u$ and $Bv = r(B)v$ for two linearly independent vectors u and v such that $e^t|u| = e^t|v| = 1$. By the arguments in the previous paragraphs, we see that there are $\theta, \phi \in \mathbb{R}$ such that $u = e^{i\theta}|u|$ and $v = e^{i\phi}|v|$, such that $|u|, |v|$ have positive entries. So, there is $\beta > 0$ such that $|u| - \beta|v|$ is nonnegative with at least one zero entry. We have $r(B)(|u| - \beta|v|) = B(|u| - \beta|v|)$, and $B(|u| - \beta|v|)$ has a positive entries, which is a contradiction. So, $|u| = |v|$.

Let x be the unique positive eigenvector such that $Bx = r(B)x$ satisfying $e^tx = 1$. Then we can consider B^t and obtain a positive vector $B^ty = r(B)y$ satisfying $x^ty = 1$. Let $S = [x|S_1] \in M_n$ be such that the columns of $y^tS_1 = [0, \dots, 0] \in \mathbb{R}^{1\times n-1}$. Then x is not in the column space of S_1 because $y^tx = 1 \ne 0$. So, S is invertible. Moreover, $y^tS = [1, 0, \dots, 0]$ so that y is the first row of S^{-1}. Now, if $S^{-1}BS = C$, then $SC = BS$ has first column equal $r(B)e_1$. Thus, the first column of C is $r(B)e_1$. Similarly, the first column of $CS^{-1} = S^{-1}B$ equals $r(B)y^t$. Thus, the first row of C is $r(B)e_1^t$. Hence, $S^{-1}BS = [r(B)] \oplus B_1$ such that $r(B_1) < r(B)$. Assertion 2 follows.

Assertion 3 *The conclusion of Theorem 5.3.1 holds.*

Proof. Note that the vectors x and y in Assertion 3 are the left and right eigenvectors of A corresponding to a simple eigenvalue λ of A with $|\lambda| = r(A)$. Now, $Ax = \lambda x$ implies that $\lambda = r(A)$. So, $S^{-1}AS = [r(A)] \oplus A_1$ such that $r(A_1) < r(A)$. Finally,

$$\lim_{m\to\infty}[A/r(A)]^m = \lim_{m\to\infty} S([1] \oplus (A_1/r(A))^m)S^{-1} = S([1] \oplus 0_{n-1})S^{-1} = xy^t.$$ □

In general, for any nonnegative matrix $A \in M_n$, we can consider $A_\varepsilon = A + \varepsilon ee^t$ for some positive $\varepsilon > 0$ so that the resulting matrix is positive so that $r(A_\varepsilon)$ is a simple eigenvalue of A_ε) with positive left and right eigenvectors x_ε and y_ε. By continuity, we have the following.

Corollary 5.3.3 *Let $A \in M_n$ be a nonnegative matrix. Then $r(A)$ is an eigenvalue of A with at least one pair of nonnegative left and right eigenvector.*

For a nonnegative matrix A, $r(A)$ is call the Perron eigenvalue of A, and the corresponding nonnegative left and right eigenvectors are called the Perron eigenvectors.

Example 5.3.4 *Note that A^k is not positive for any positive integer k in all the following.*

If $A = I_2$, then $r(A) = 1$ and all nonzero vectors are left and right eigenvectors.

If $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, then $r(A) = 1$ with right and left eigenvectors $x = (1, 0)^t/2$ and $y = (0, 1)^t$.

If $A = \begin{pmatrix} 1/2 & 1/2 \\ 0 & 1 \end{pmatrix}$, then $r(A) = 1$ with right and left eigenvectors $x = (1, 1)^t/2$ and $y = (0, 2)^t$.

If $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then $r(A) = 1$ with right and left eigenvectors $x = (1, 1)^t/2$ $y = (1, 1)^t$.

A row (column) stochastic matrix is a matrix with nonnegative entries such that all row (colum) sums equal one. It appear in the study of Markov Chain in probability, population models, Google page rank matrix, etc. If $A \in M_n$ is both row and column stochastic, then it is doubly stochastic.

Corollary 5.3.5 *Let A be a row stochastic matrix. Then $r(A) = 1$. If A^k is positive, then $r(A)$ is a simple eigenvalue with a unique positive left eigenvector x satisfying $e^t x = 1$, and a unique positive left eigenvector y such that $A^k \to xy^t$ as $k \to \infty$.*

5.4 Kronecker (tensor) products

Definition 5.4.1 *Let $A = (a_{ij}) \in M_{m,n}, B = (b_{rs}) \in M_{p,q}$. Then $A \otimes B = (a_{ij}B) \in M_{mp,ns}$.*

Theorem 5.4.2 *The following equations hold for scalar a, b and matrices A, B, C, D) provided that the sizes of the matrices are compatible with the described operations.*

(a) $(aA + bB) \otimes C = aA \otimes C + bB \otimes C$, $C \otimes (aA + bB) = aC \otimes A + bC \otimes B$.

(b) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.

Proof. (a) By direct verification. (b) Suffices to show $(A \otimes B)(C_j \otimes D_k) = (AC_j) \otimes (BD_k)$ for all columns C_j of C and D_k of D. □

Corollary 5.4.3 *Let A, B be matrices. Then $f(A \otimes B) = f(A) \otimes f(B)$ for $f(X) = \overline{X}, X^t$ or X^*.*

(a) *If A, B are invertible, then $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.*

(b) *If A and B are unitary, then so is $A \otimes B$ with inverse $(A \otimes B)^* = A^* \otimes B^*$.*

(c) *If* $S^{-1}AS$ *and* $T^{-1}BS$ *are in triangular forms, then so is* $(S \otimes T)^{-1}(A \otimes B)(S \otimes T)$.

(d) *If* A *has eigenvalues* $a_1, \dots, a_m$ *and* B *has eigenvalues* $b_1, \dots, b_n$, *then* $A \otimes B$ *has eigenvalues* $a_i b_j$ *with* $1 \le i \le m, 1 \le j \le n$; *if* x_i, y_j *are eigenvectors such that* $Ax_i = a_i x_i$ *and* $By_j = b_j y_j$, *then* $(A \otimes B)(x_i \otimes j) = a_i b_j (x_i \otimes y_j)$.

(e) *If* A *and* B *have singular value decomposition* $A = U_1 D_1 V_1^*$ *and* $B = U_2 D_2 V_2^*$, *then the equation* $(A \otimes B)(V_1 \otimes V_2) = (U_1 \otimes U_2)(D_1 \otimes D_2)$ *will yield the information for singular values and singular vectors.*

We have the following application of the tensor product results to matrix equations.

Theorem 5.4.4 *Let* $A \in M_m, B \in M_n$ *and* $C \in M_{m,n}$. *Then the matrix equation*

$$AX + XB = C \qquad X \in M_{m,n}$$

can be rewritten as $(I_m \otimes A)\text{vec}(X) + (B^t \otimes I_n)\text{vec}(X) = \text{vec}(C)$, *where for* $Z \in M_{m,n}$ *we have* $\text{vec}(Z) \in \mathbb{C}^{mn}$ *with the first column of* Z *as the first* m *entries, second column of* Z *as the next* m *entries, etc.*

Consequently, the matrix equation is solvable if and only if $\text{vec}(C)$ *lies in the column space of* $I_n \otimes A + B^t \otimes I_m$. *In particular, if* $I_n \otimes A + B^t \otimes I_m$ *is invertible, then the matrix equation is always solvable.*

The Hadamard (Schur) product of two matrices $A = (a_{ij}), B = (b_{ij}) \in M_{m,n}$ is defined by $A \circ B = (a_{ij} b_{ij})$.

Corollary 5.4.5 *Let* $A, B \in M_{m,n}$.

(a) *Then* $s_k(A \otimes B) \ge s_k(A \circ B)$ *for* $k = 1, \dots, m$.

(b) *If* $m = n$, *then* $s_{n-k+1}(A \circ B) \ge s_{n^2-k+1}(A \otimes B)$ *for* $k = 1, \dots, n$.

(c) *If* A, B *are positive semidefinite, then so is* $A \circ B$.

Remark Note that if $A, B \in M_n$ are invertible, unitary, or normal, it does not follow that $A \circ B$ has the same property.

5.5 Compound matrices

Let $A \in M_{m,n}$ and $k \le \min\{m, n\}$. Then the compound matrix $C_m(A)$ is of size $\binom{m}{k} \times \binom{n}{k}$ with rows labeled by increasing subseqeuence $r = (r_1, \dots, r_k)$ of $(1, \dots, m)$ and columns labeled by increasing subseqeuence $s = (s_1, \dots, s_k)$ of $(1, \dots, n)$ in lexicographic order such that the (r, s) entry of $C_m(A)$ equals $\det(A[r, s])$, where $A[r, s] \in M_k$ is the submatrix of A with rows and columns indexed r and s, arranged in lexicographic order.

Example 5.5.1 *Let $A \in M_4$. Then $C_2(A) \in M_6$ with $(r_1, r_2), (s_1, s_2)$ entry equal to* $\det(A[r_1, r_2; s_1, s_2])$.

It is easy to check that $C_k(A^t) = C_k(A)^t$, $C_k(A^*) = C_k(A)^*$, etc.

We will prove a product formula for the compound matrix. The proof depends on the following result which generalizes the Cauchy-Binet formula.

Theorem 5.5.2 *Let $A \in M_{m,n}$ and $B \in M_{n,m}$. Then for any $1 \le k \le m$, the sum of the $k \times k$ principal minors of AB is the same as that of $BA \in M_n$.*

Note that when $k = m \le n$, the above result is known as the Cauchy Binet formula.

Proof. Recall that if

$$P = \begin{pmatrix} AB & 0 \\ B & 0_n \end{pmatrix}, \quad Q = \begin{pmatrix} 0_m & 0 \\ B & BA \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix},$$

then S is invertible and

$$PS = \begin{pmatrix} AB & 0 \\ B & 0_n \end{pmatrix} \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} AB & ABA \\ B & BA \end{pmatrix} = \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} \begin{pmatrix} 0_m & 0 \\ B & BA \end{pmatrix} = SQ$$

Thus, P and Q are similar, and

$$z^m \det(zI_n - BA) = \det(zI_{m+n} - Q) = det(zI_{m+n} - P) = z^n \det(zI_m - AB).$$

Thus the sum of the kth principal minors of P and that of Q are the same. Evidently, the sum of the kth principal minors of P are the same as that of AB, and the sum of the kth principal minors of Q are the same as that of BA. The result follows. □

Theorem 5.5.3 *Let $A \in M_{m,n}$, $B \in M_{n,p}$ and $k \le \min\{m, n, p\}$. Then $C_k(AB) = C_k(A)C_k(B)$.*

Proof. Let $\Gamma_{r,k}$ be the set of length k increasing subsequence of $(1, \dots, r)$ for $r \ge k$. Consider the entry of $C_k(AB)$ with row indexes $r = (r_1, \dots, r_k) \in \Gamma_{m,k}$ and column indexes $s = (s_1, \dots, s_k) \in \Gamma_{n,k}$. Let $\hat{A} \in M_{k,n}$ be obtained from A by using its rows indexed by $(r_1, \dots, r_k)$, and let $\hat{B} \in M_{n,k}$ be obtained from B by using its columns indexed by $(s_1, \dots, s_k)$. Then the (r, s) entry of $C_k(AB)$ equals $\det(\hat{A}\hat{B}) = C_k(\hat{A})C_k(\hat{B})$ by the Cauchy Binet formula. Note that $C_k(\hat{A})C_k(\hat{B})$ is the (r, s) entry of $C_k(A)C_k(B)$. The result follows. □

Corollary 5.5.4 *Let $A \in M_n$ and $k \le n$.*

(a) *If A is invertible (unitary), then so is $C_k(A)$.*

(b) *Suppose $A = UTU^*$ is in triangular form. Then $C_k(A) = C_k(U)C_k(T)C_k(U^*)$, where $C_k(T)$ is in triangular form. Consequently, $C_k(A)$ has eigenvalues $\prod_{j=1}^k \lambda_{i_j}(A)$.*

(c) *Suppose* $U^*AV = D$ *with* $D = \sum_{j=1}^n s_j(A)E_{jj}$, *where* U, V *are unitary. Then* $C_k(U^*)C_k(A)C_k(V) = C_k(D)$. *Consequently,* $C_k(A)$ *has singular values* $\prod_{j=1}^k s_{i_j}(A)$, $1 \le i_1 < \cdots < i_k \le n$.

Let $A \in M_n$ and $1 \le k \le n$, and

$$C_k(tI_n + A) = C_k(A) + tD_k(A) + t^2 D_{2,k} + t^{k-3} D_{3,k}(A) + \cdots + t^k I.$$

The matrix $D_k(A)$ is called the additive compound of A.

Theorem 5.5.5 *Let* $A \in M_n$. *Then* $D_k(S^{-1}AS) = C_k(S)^{-1}D_k(A)C_k(S)$ *so that* A *has eigenvalues* $\sum_{j=1}^k \lambda_{i_j}(A)$ *with* $1 \le i_1 < \cdots < i_k \le n$. *Consequently, if* A *is normal (Hermitian, positive semi-definite) then so is* $D_k(A)$.

Theorem 5.5.6 *Let* $A, B \in M_n$. *Then* $D_k(AB) = D_k(AB - BA) = D_k(A)D_k(B) - D_k(B)D_k(A)$. *Consequently, if* A *and* B *commute, then so do* $D_k(A)$ *and* $D_k(B)$.

Proof. The proof follows from the fact that $D_k(X)$ can be written as

$$V^* \left(\sum_{j=1}^k (\underbrace{I_n \otimes \cdots \otimes I_n}_{j-1} \otimes X \otimes \underbrace{I_n \otimes \cdots \otimes I_n}_{k-j} \right) V,$$

where $V \in M_{n^k \times \binom{n}{k}}$ such that $V^*V = I_{\binom{n}{k}}$ and the columns of V is a basis for the subspace of $\mathbb{C}^{n^k}$ spanned by

$$\left\{ \sum_{\sigma \in S_k} \chi(\sigma) e_{\sigma(i_1)} \otimes \cdots \otimes e_{\sigma(i_k)} : 1 \le i_1 < \cdots < i_k \le n \right\},$$

where $\chi(\sigma) = 1$ if $\sigma \in S_k$ is an even permutation and $\chi(\sigma) = -1$ otherwise. □

5.6 More block matrix techniques

Schur Complement.

Block PSD matrices.