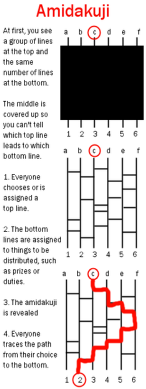
Factorization of permutation

Chi-Kwong Li
Department of Mathematics
College of William and Mary

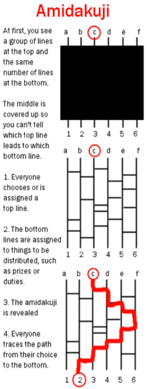
Based on the paper:
Zejun Huang, Chi-Kwong Li, Sharon H. Li, Nung-Sing Sze,

Amidakuji/Ghost Leg Drawing



Amidakuji/Ghost Leg Drawing

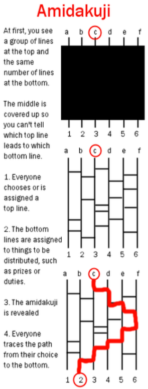
It is a scheme for assigning n people $P_1, \ldots, P_n$ to n jobs $J_1, \ldots, J_n$ "randomly".



Amidakuji/Ghost Leg Drawing

It is a scheme for assigning n people $P_1, \ldots, P_n$ to n jobs $J_1, \ldots, J_n$ "randomly".

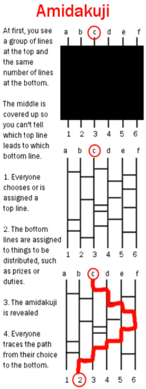
- Draw vertical lines from P_i to J_i from $i = 1, \ldots, n$.



Amidakuji/Ghost Leg Drawing

It is a scheme for assigning n people $P_1, \dots, P_n$ to n jobs $J_1, \dots, J_n$ "randomly".

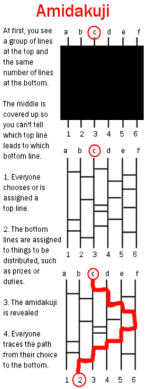
- Draw vertical lines from P_i to J_i from $i = 1, \dots, n$.
- Draw some horizontal line segments randomly between any two vertical lines that are next to each other so that no horizontal lines meet.



Amidakuji/Ghost Leg Drawing

It is a scheme for assigning n people $P_1, \ldots, P_n$ to n jobs $J_1, \ldots, J_n$ "randomly".

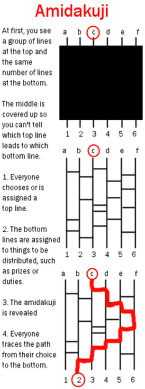
- Draw vertical lines from P_i to J_i from $i = 1, \ldots, n$.
- Draw some horizontal line segments randomly between any two vertical lines that are next to each other so that no horizontal lines meet.
- To assign a job for P_i, start from the top of the i-th line to the bottom, and make a turn whenever a horizontal segment is encountered.



Amidakuji/Ghost Leg Drawing

It is a scheme for assigning n people $P_1, \ldots, P_n$ to n jobs $J_1, \ldots, J_n$ "randomly".

- Draw vertical lines from P_i to J_i from $i = 1, \ldots, n$.
- Draw some horizontal line segments randomly between any two vertical lines that are next to each other so that no horizontal lines meet.
- To assign a job for P_i, start from the top of the i-th line to the bottom, and make a turn whenever a horizontal segment is encountered.



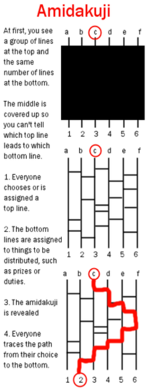
Questions

- Why do we always get an one-one correspondence (bijection)?

Amidakuji/Ghost Leg Drawing

It is a scheme for assigning n people $P_1, \ldots, P_n$ to n jobs $J_1, \ldots, J_n$ "randomly".

- Draw vertical lines from P_i to J_i from $i = 1, \ldots, n$.
- Draw some horizontal line segments randomly between any two vertical lines that are next to each other so that no horizontal lines meet.
- To assign a job for P_i, start from the top of the i-th line to the bottom, and make a turn whenever a horizontal segment is encountered.



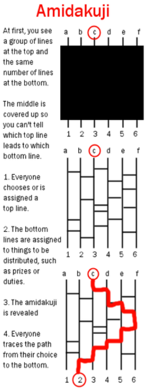
Questions

- Why do we always get an one-one correspondence (bijection)?
- Can we get all possible job assignments?

Amidakuji/Ghost Leg Drawing

It is a scheme for assigning n people $P_1, \ldots, P_n$ to n jobs $J_1, \ldots, J_n$ "randomly".

- Draw vertical lines from P_i to J_i from $i = 1, \ldots, n$.
- Draw some horizontal line segments randomly between any two vertical lines that are next to each other so that no horizontal lines meet.
- To assign a job for P_i, start from the top of the i-th line to the bottom, and make a turn whenever a horizontal segment is encountered.



Questions

- Why do we always get an one-one correspondence (bijection)?
- Can we get all possible job assignments?
- What is the minimum number of horizontal segments needed for a given job assignment?

Answer of Question 1

George Polya (1887-1985)

If one cannot solve a problem,
one can try to solve an easier problem first.

Answer of Question 1

George Polya (1887-1985)

If one cannot solve a problem,
one can try to solve an easier problem first.

- What if there is no horizontal line segment?

George Polya (1887-1985)

If one cannot solve a problem,
one can try to solve an easier problem first.

- What if there is no horizontal line segment?
- What if there is one horizontal line segment?

George Polya (1887-1985)

If one cannot solve a problem,
one can try to solve an easier problem first.

- What if there is no horizontal line segment?
- What if there is one horizontal line segment?
- An easy induction argument!

Bubble sort

- Regard the job assignment as a permutation (a seat assignment)

$$\sigma = [i_1, \ldots, i_n] = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix}.$$

Bubble sort

- Regard the job assignment as a permutation (a seat assignment)

$$\sigma = [i_1, \ldots, i_n] = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix}.$$

- Use Coxeter transpositions $(i, i+1)$ for $i = 1, \ldots, n-1$.

Bubble sort

- Regard the job assignment as a permutation (a seat assignment)

$$\sigma = [i_1, \ldots, i_n] = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix}.$$

- Use Coxeter transpositions $(i, i+1)$ for $i = 1, \ldots, n-1$.
- For any σ, we can determine the total number $\iota(\sigma)$ of inversions of σ.

Bubble sort

- Regard the job assignment as a permutation (a seat assignment)

$$\sigma = [i_1, \ldots, i_n] = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix}.$$

- Use Coxeter transpositions $(i, i+1)$ for $i = 1, \ldots, n-1$.
- For any σ, we can determine the total number $\iota(\sigma)$ of inversions of σ.
- It is the minimum number of Coxeter transpositions needed to generate σ.

Bubble sort

- Regard the job assignment as a permutation (a seat assignment)
$$\sigma = [i_1, \ldots, i_n] = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix}.$$
- Use Coxeter transpositions $(i, i+1)$ for $i = 1, \ldots, n-1$.
- For any σ, we can determine the total number $\iota(\sigma)$ of inversions of σ.
- It is the minimum number of Coxeter transpositions needed to generate σ.

Example For $\sigma = [5, 3, 1, 2, 4]$, total number of inversions is: $4 + 0 + 2 = 6$, and

$$\sigma \to [3, 5, 1, 2, 4] \to [3, 1, 5, 2, 4] \to [3, 1, 2, 5, 4]$$
$$\to [3, 1, 2, 4, 5] \to [1, 3, 2, 4, 5] \to [1, 2, 3, 4, 5],$$

Bubble sort

- Regard the job assignment as a permutation (a seat assignment)

$$\sigma = [i_1, \ldots, i_n] = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix}.$$

- Use Coxeter transpositions $(i, i+1)$ for $i = 1, \ldots, n-1$.
- For any σ, we can determine the total number $\iota(\sigma)$ of inversions of σ.
- It is the minimum number of Coxeter transpositions needed to generate σ.

Example For $\sigma = [5,3,1,2,4]$, total number of inversions is: $4+0+2=6$, and

$$\sigma \to [3,5,1,2,4] \to [3,1,5,2,4] \to [3,1,2,5,4]$$

$$\to [3,1,2,4,5] \to [1,3,2,4,5] \to [1,2,3,4,5],$$

So $\sigma = (1,2)(2,3)(3,4)(4,5)(1,2)(2,3)$.

Bubble sort

- Regard the job assignment as a permutation (a seat assignment)

$$\sigma = [i_1, \ldots, i_n] = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix}.$$

- Use Coxeter transpositions $(i, i+1)$ for $i = 1, \ldots, n-1$.
- For any σ, we can determine the total number $\iota(\sigma)$ of inversions of σ.
- It is the minimum number of Coxeter transpositions needed to generate σ.

Example For $\sigma = [5, 3, 1, 2, 4]$, total number of inversions is: $4+0+2=6$, and

$$\sigma \to [3,5,1,2,4] \to [3,1,5,2,4] \to [3,1,2,5,4]$$
$$\to [3,1,2,4,5] \to [1,3,2,4,5] \to [1,2,3,4,5],$$

So $\sigma = (1,2)(2,3)(3,4)(4,5)(1,2)(2,3)$.

Answers of Questions 2 and 3

- We can always convert a permutation σ to $[1, \ldots, n]$ using $\iota(\sigma)$ steps, where $\iota(\sigma)$ is the number of inversions of σ.

Bubble sort

- Regard the job assignment as a permutation (a seat assignment)

$$\sigma = [i_1, \ldots, i_n] = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix}.$$

- Use Coxeter transpositions $(i, i+1)$ for $i = 1, \ldots, n-1$.
- For any σ, we can determine the total number $\iota(\sigma)$ of inversions of σ.
- It is the minimum number of Coxeter transpositions needed to generate σ.

Example For $\sigma = [5, 3, 1, 2, 4]$, total number of inversions is: $4 + 0 + 2 = 6$, and

$$\sigma \to [3, 5, 1, 2, 4] \to [3, 1, 5, 2, 4] \to [3, 1, 2, 5, 4]$$
$$\to [3, 1, 2, 4, 5] \to [1, 3, 2, 4, 5] \to [1, 2, 3, 4, 5],$$

So $\sigma = (1, 2)(2, 3)(3, 4)(4, 5)(1, 2)(2, 3)$.

Answers of Questions 2 and 3

- We can always convert a permutation σ to $[1, \ldots, n]$ using $\iota(\sigma)$ steps, where $\iota(\sigma)$ is the number of inversions of σ.
- **Worst case** occurs at $[n, n-1, \ldots, 1]$; which requires

Bubble sort

- Regard the job assignment as a permutation (a seat assignment)

$$\sigma = [i_1, \ldots, i_n] = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix}.$$

- Use Coxeter transpositions $(i, i+1)$ for $i = 1, \ldots, n-1$.
- For any σ, we can determine the total number $\iota(\sigma)$ of inversions of σ.
- It is the minimum number of Coxeter transpositions needed to generate σ.

Example For $\sigma = [5, 3, 1, 2, 4]$, total number of inversions is: $4 + 0 + 2 = 6$, and

$$\sigma \to [3, 5, 1, 2, 4] \to [3, 1, 5, 2, 4] \to [3, 1, 2, 5, 4]$$
$$\to [3, 1, 2, 4, 5] \to [1, 3, 2, 4, 5] \to [1, 2, 3, 4, 5],$$

So $\sigma = (1, 2)(2, 3)(3, 4)(4, 5)(1, 2)(2, 3)$.

Answers of Questions 2 and 3

- We can always convert a permutation σ to $[1, \ldots, n]$ using $\iota(\sigma)$ steps, where $\iota(\sigma)$ is the number of inversions of σ.
- **Worst case** occurs at $[n, n-1, \ldots, 1]$; which requires

$$(n-1) + \cdots + 1 = n(n-1)/2 \text{ steps.}$$

A variation of Amidakuji

- What if we consider transpositions of the forms $(i, i+1)$ and $(i, i+2)$?

A variation of Amidakuji

- What if we consider transpositions of the forms $(i, i+1)$ and $(i, i+2)$?
- How about using transpositions $(i, i+1), (i, i+2), (i, i+3)$, etc.?

A variation of Amidakuji

- What if we consider transpositions of the forms $(i, i+1)$ and $(i, i+2)$?
- How about using transpositions $(i, i+1), (i, i+2), (i, i+3)$, etc.?

An extreme case: Using all (i, j) with $1 \le j < n$

Decompose σ as product of k disjoint cycles (including fixed points).

- What if we consider transpositions of the forms $(i, i+1)$ and $(i, i+2)$?
- How about using transpositions $(i, i+1), (i, i+2), (i, i+3)$, etc.?

An extreme case: Using all (i, j) with $1 \leq j < n$

Decompose σ as product of k disjoint cycles (including fixed points).

Then σ is a product of $n - k$ transpositions.

A variation of Amidakuji

- What if we consider transpositions of the forms $(i, i+1)$ and $(i, i+2)$?
- How about using transpositions $(i, i+1), (i, i+2), (i, i+3)$, etc.?

An extreme case: Using all (i, j) with $1 \leq j < n$

Decompose σ as product of k disjoint cycles (including fixed points).

Then σ is a product of $n - k$ transpositions.

So, the worst case requires $n - 1$ steps.

A variation of Amidakuji

- What if we consider transpositions of the forms $(i, i+1)$ and $(i, i+2)$?
- How about using transpositions $(i, i+1), (i, i+2), (i, i+3)$, etc.?

An extreme case: Using all (i, j) with $1 \leq j < n$

Decompose σ as product of k disjoint cycles (including fixed points).

Then σ is a product of $n-k$ transpositions.

So, the worst case requires $n-1$ steps.

Example. $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 4 & 5 & 9 & 6 & 7 & 1 & 8 & 2 \end{pmatrix} = (1,3,5,6,7)(2,4,9)(8).$

A variation of Amidakuji

- What if we consider transpositions of the forms $(i, i+1)$ and $(i, i+2)$?
- How about using transpositions $(i, i+1), (i, i+2), (i, i+3)$, etc.?

An extreme case: Using all (i, j) with $1 \leq j < n$

Decompose σ as product of k disjoint cycles (including fixed points).

Then σ is a product of $n - k$ transpositions.

So, the worst case requires $n - 1$ steps.

Example. $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 4 & 5 & 9 & 6 & 7 & 1 & 8 & 2 \end{pmatrix} = (1,3,5,6,7)(2,4,9)(8).$

Then $\sigma = (1,7)(1,6)(1,5)(1,3)(2,9)(2,4)$.

Some open problems

Let $1 \leq m < n$, and let G_m be the set of transpositions of the form $(i, i+\ell)$ with $1 \leq \ell \leq m$.

Some open problems

Let $1 \leq m < n$, and let G_m be the set of transpositions of the form $(i, i + \ell)$ with $1 \leq \ell \leq m$.

- For a given $\sigma \in S_n$, find the smallest r such that σ is the product of r transpositions in G_m.

Some open problems

Let $1 \leq m < n$, and let G_m be the set of transpositions of the form $(i, i+\ell)$ with $1 \leq \ell \leq m$.

- For a given $\sigma \in S_n$, find the smallest r such that σ is the product of r transpositions in G_m.
- Determine the optimal (smallest) $r^* = r^*(n, m)$ so that every $\sigma \in S_n$ is a product at most r^* transpositions in G_m.

Some open problems

Let $1 \leq m < n$, and let G_m be the set of transpositions of the form $(i, i+\ell)$ with $1 \leq \ell \leq m$.

- For a given $\sigma \in S_n$, find the smallest r such that σ is the product of r transpositions in G_m.
- Determine the optimal (smallest) $r^* = r^*(n, m)$ so that every $\sigma \in S_n$ is a product at most r^* transpositions in G_m.
- To find r^* and the permutation which is most difficult to get restore, we use the breadth first search.

Partial results of the general problem

We have the following list for $r^*(n,m)$ for S_n and $(i, i+\ell)$ with $\ell \leq m$,

$n\backslash m$	1	2	3	4	5	6	7	8	9	10	11
2	1										
3	3	2									
4	6	4	3								
5	10	5	5	4							
6	15	[7]	6	6	5						
7	21	[10]	8	7	7	6					
8	28	[14]	[10]	9	8	8	7				
9	36	[16]	[11]	10	10	9	9	8			
10	45	[19]	[14]	[12]	11	11	10	10	9		
11	55	[23]	[16]	[14]	13	12	12	11	11	10	
12	66	29^*	20^*	17^*	16^*	14	13	13	12	12	11

where the entries marked by brackets are obtained by computer programming.

Another variation (The round table version)

Theorem [Jerrum, 1985], [van Zuylen et. al, 2014]

Only use transpositions: $(n, 1)$ and $(i, i+1) : i = 1, \ldots, n-1$.

Another variation (The round table version)

Theorem [Jerrum, 1985], [van Zuylen et. al, 2014]

Only use transpositions: $(n, 1)$ and $(i, i+1) : i = 1, \dots, n-1$.
Given a permutation $[p_1, \dots, p_n]$.

Another variation (The round table version)

Theorem [Jerrum, 1985], [van Zuylen et. al, 2014]

Only use transpositions: $(n, 1)$ and $(i, i+1) : i = 1, \ldots, n-1$.
Given a permutation $[p_1, \ldots, p_n]$.

- let $d = [d_1, \ldots, d_n] = [p_1, \cdots, p_n] - [1, \ldots, n]$ so that $\sum d_i = 0$;

Another variation (The round table version)

Theorem [Jerrum, 1985], [van Zuylen et. al, 2014]

Only use transpositions: $(n, 1)$ and $(i, i+1) : i = 1, \ldots, n-1$.
Given a permutation $[p_1, \ldots, p_n]$.

- let $d = [d_1, \ldots, d_n] = [p_1, \cdots, p_n] - [1, \ldots, n]$ so that $\sum d_i = 0$;
- modify d to $\tilde{d}$ by replacing (d_i, d_j) by $(d_i - n, n + d_j)$ if $d_i - d_j > n$ until $p_r - p_s \le n$ for all r, s.

Another variation (The round table version)

Theorem [Jerrum, 1985], [van Zuylen et. al, 2014]

Only use transpositions: $(n,1)$ and $(i,i+1) : i = 1,\ldots,n-1$.
Given a permutation $[p_1,\ldots,p_n]$.

- let $d = [d_1,\ldots,d_n] = [p_1,\cdots,p_n] - [1,\ldots,n]$ so that $\sum d_i = 0$;
- modify d to $\tilde{d}$ by replacing (d_i, d_j) by $(d_i - n, n + d_j)$ if $d_i - d_j > n$ until $p_r - p_s \le n$ for all r, s.
- Restore the permutation using this displacement vector $\tilde{d}$ will use the minimum number of steps $\iota(\tilde{d})$,

Another variation (The round table version)

Theorem [Jerrum, 1985], [van Zuylen et. al, 2014]

Only use transpositions: $(n,1)$ and $(i,i+1): i = 1,\dots,n-1$.
Given a permutation $[p_1,\dots,p_n]$.

- let $d = [d_1,\dots,d_n] = [p_1,\cdots,p_n] - [1,\dots,n]$ so that $\sum d_i = 0$;
- modify d to $\tilde{d}$ by replacing (d_i,d_j) by $(d_i - n, n + d_j)$ if $d_i - d_j > n$ until $p_r - p_s \le n$ for all r,s.
- Restore the permutation using this displacement vector $\tilde{d}$ will use the minimum number of steps $\iota(\tilde{d})$, which is the generalized inversion number of $\tilde{p} = \tilde{d} + [1,\dots,n]$.

Another variation (The round table version)

Theorem [Jerrum, 1985], [van Zuylen et. al, 2014]

Only use transpositions: $(n,1)$ and $(i,i+1): i=1,\dots,n-1$.
Given a permutation $[p_1,\dots,p_n]$.

- let $d=[d_1,\dots,d_n]=[p_1,\cdots,p_n]-[1,\dots,n]$ so that $\sum d_i=0$;
- modify d to $\tilde{d}$ by replacing (d_i,d_j) by $(d_i-n,n+d_j)$ if $d_i-d_j>n$ until $p_r-p_s\le n$ for all r,s.
- Restore the permutation using this displacement vector $\tilde{d}$ will use the minimum number of steps $\iota(\tilde{d})$, which is the generalized inversion number of $\tilde{p}=\tilde{d}+[1,\dots,n]$.

The number of steps is at most $[n^2/4]$ attained at the following permutation:

(1) $[k+1,\dots,n,1,\dots,k]$ if $n=2k$ or $n=2k+1$,

(2) $[k+2,\dots,n,1,\dots,k+1]$ or $[k+1,\dots,n,1,\dots,k]$ if $n=2k+1$.

Another variation (The round table version)

Theorem [Jerrum, 1985], [van Zuylen et. al, 2014]

Only use transpositions: $(n,1)$ and $(i,i+1) : i = 1,\ldots,n-1$.
Given a permutation $[p_1,\ldots,p_n]$.

- let $d = [d_1,\ldots,d_n] = [p_1,\cdots,p_n] - [1,\ldots,n]$ so that $\sum d_i = 0$;
- modify d to $\tilde{d}$ by replacing (d_i,d_j) by $(d_i - n, n + d_j)$ if $d_i - d_j > n$ until $p_r - p_s \leq n$ for all r,s.
- Restore the permutation using this displacement vector $\tilde{d}$ will use the minimum number of steps $\iota(\tilde{d})$, which is the generalized inversion number of $\tilde{p} = \tilde{d} + [1,\ldots,n]$.

The number of steps is at most $[n^2/4]$ attained at the following permutation:

(1) $[k+1,\ldots,n,1,\ldots,k]$ if $n = 2k$ or $n = 2k+1$,

(2) $[k+2,\ldots,n,1,\ldots,k+1]$ or $[k+1,\ldots,n,1,\ldots,k]$ if $n = 2k+1$.

Example $p = [6,5,1,2,4,3]$, $d = [5,3,-2,-2,-1,-3]$,

$$\tilde{d} = [-1,3,-2,-2,-1,3],\ \tilde{p} = [0,5,1,2,4,9],\ \iota(\tilde{d}) = 6,$$

Note For $[4,5,6,1,2,3]$, $d = [3,3,3,-3,-3,-3]$ and $\iota(d) = 9 = [6^2/4]$.

Some open problems

- What if we can use the the circular permutations (i, j) with $|j - i| \leq k$ with $k = 1, 2, \ldots$ in the round table problem?

Some open problems

- What if we can use the the circular permutations (i, j) with $|j - i| \leq k$ with $k = 1, 2, \ldots$ in the round table problem?
- One can use $L = (1, 2, \ldots, n)$ and $S = (1, 2)$ to generate all permutations. Then the maximum steps needed are:

S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	S_{12}
1	2	6	11	18	25	35	45	58	71	???

Some open problems

- What if we can use the the circular permutations (i, j) with $|j - i| \leq k$ with $k = 1, 2, \ldots$ in the round table problem?
- One can use $L = (1, 2, \ldots, n)$ and $S = (1, 2)$ to generate all permutations. Then the maximum steps needed are:

S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	S_{12}
1	2	6	11	18	25	35	45	58	71	???

- If we use L, S, L^{-1}, then the maximum steps needed are:

S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}
1	2	6	10	15	21	28	36	45	???

Conjecture We need at most $\binom{n}{2}$ steps, and the worst case is

$$[2, 1, n, n-1, \ldots, 3].$$

Related research

- Theoretical computer science?

Related research

- Theoretical computer science? Determine the optimal sorting algorithm with the given operations, and determine the worst scenario.

Related research

- Theoretical computer science? Determine the optimal sorting algorithm with the given operations, and determine the worst scenario.
- The study of genomics and mutations,

Related research

- Theoretical computer science? Determine the optimal sorting algorithm with the given operations, and determine the worst scenario.
- The study of genomics and mutations, i.e., the change of genetic sequences $x_1x_2x_3\cdots$, with $x_i \in \{A, U, G, C\}$.

Related research

- Theoretical computer science? Determine the optimal sorting algorithm with the given operations, and determine the worst scenario.
- The study of genomics and mutations, i.e.,
  the change of genetic sequences $x_1x_2x_3\cdots$, with $x_i \in \{A, U, G, C\}$.
- Quantum computing.
  It is of interest to decompose certain quantum gates into simpler quantum gates (CNOT gates).

Let me know if you have any thought!

Let me know if you have any thought!

Thank you for your attention!