Quantum States and Quantum Channels

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where $F_1, \ldots, F_r$ are $m \times n$ matrices satisfying $F_1^* F_1 + \cdots + F_r^* F_r = I_n$.

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- Evidently, if $C(\Phi) = (Q_{ij})$, then $\Phi(\rho) = \sum \rho_{ij} Q_{ij}$.

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- It is convenient to apply a permutation similarity, and consider

$$\mathcal{UC}(2,n) = \left\{P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \in \mathcal{M}_2(\mathcal{M}_n) : P \text{ is PSD}, \ P_{11} + P_{22} = I_n \right\},$$

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- Note that $P \in \mathcal{UC}(2,n)$ if and only if $\rho = \frac{1}{n}P$ is a quantum states in the bipartite system $\mathcal{M}_2 \otimes \mathcal{M}_n$ such that $\mathrm{tr}_1\rho = \frac{1}{n}I_n$, the maximally mixed state.

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(c) (c.1) $P + (H \oplus -H)$ is not PSD if $H = H^* \neq 0$,

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So, the convex sets $\mathcal{UC}(2,n)$ and $\mathcal{R}$ have the same structure.

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If k satisfies $n/2 \le k \le n$, then for $n = k + s$

$$P = I_{k-s} \oplus \begin{pmatrix} I_s & & & I_s \\ & 0_{n-k} & & \\ & & 0_{n-s} & \\ I_s & & & I_s \end{pmatrix}$$

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If $n < k \le 2n$, then for $k = n + s$

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- The lower bound of the von Neumann entropy of $Q \in \mathcal{QC}(n, 2)$ is attained at the lowest rank matrix corresponding to $P \in \mathcal{UC}(2, n)$, where

$$P = \begin{cases} \begin{pmatrix} I_k & & I_k \\ & 0_n & \\ I_k & & I_k \end{pmatrix} & \text{if } n = 2k \text{ is even} \\ [1] \oplus \begin{pmatrix} I_k & & I_k \\ & 0_{n+1} & \\ I_k & & I_k \end{pmatrix} & \text{if } n = 2k+1 \text{ is odd.} \end{cases}$$

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- In fact, the same conclusions hold for other Schur concave function on $Q \in \mathcal{Q}C(n,2)$.

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$$S_2\left(\frac{1}{n}I_n\right) = \left\{ \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{12}^* & \rho_{22} \end{pmatrix} \in \mathcal{D}_{2n} : \rho_{11} + \rho_{22} = \frac{1}{n}I_n \right\}. \qquad (1)$$

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$$\Omega_n = \left\{ (a_1, \ldots, a_n) : a_1 \geq \cdots \geq a_n \geq 0 \text{ and } \sum_{j=1}^{n} a_j = 1 \right\}. \qquad (2)$$

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$$\lambda(\xi_0) = \mathbf{a}, \quad \lambda(\xi_1) = (d_1, \ldots, d_n, 0, \ldots, 0), \text{ and}$$

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(c) There are real numbers $d_1 \geq \cdots \geq d_n$ in $[0, 1/n]$ such that

$\alpha = (d_1, \ldots, d_n, 0, \ldots, 0)$ and $\beta = (1/n - d_n, \ldots, 1/n - d_1, 0, \ldots, 0)$

satisfy the Littlewood-Richardson inequalities

$$\sum_{p \in P} \alpha_p + \sum_{q \in Q} \beta_q \geq \sum_{r \in R} a_r, \quad (P, Q, R) \in LR_{2n}.$$

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Consequently, the set $\mathcal{E}_n$ is a polytope in Ω_{2n}.

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Theorem (for $\mathcal{E}_2$)

The convex set $\mathcal{E}_2$ and Ω_4 are equal with extreme points:

$$\left(1,0,0,0\right),\left(\tfrac{1}{2},\tfrac{1}{2},0,0\right),\left(\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3},0\right),\left(\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{4}\right).$$

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Theorem (for $\mathcal{E}_3$)

The set $\mathcal{E}_3$ consists of $(a_1,\ldots,a_6) \in \Omega_6$ satisfying

$$a_4 + a_5 \leq \frac{1}{3} \leq a_2 + a_3.$$

The extreme points of the sets are:

$\left(\tfrac{1}{2},\tfrac{1}{2},0,0,0,0\right)$, $\left(\tfrac{2}{3},\tfrac{1}{3},0,0,0,0\right)$, $\left(\tfrac{2}{3},\tfrac{1}{6},\tfrac{1}{6},0,0,0\right)$,
$\left(\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3},0,0,0\right)$, $\left(\tfrac{1}{2},\tfrac{1}{6},\tfrac{1}{6},\tfrac{1}{6},0,0\right)$, $\left(\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{4},0,0\right)$,
$\left(\tfrac{1}{3},\tfrac{1}{6},\tfrac{1}{6},\tfrac{1}{6},\tfrac{1}{6},0\right)$, $\left(\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{6},\tfrac{1}{6},\tfrac{1}{6},0\right)$, $\left(\tfrac{2}{9},\tfrac{2}{9},\tfrac{2}{9},\tfrac{1}{6},\tfrac{1}{6},0\right)$,
$\left(\tfrac{2}{9},\tfrac{2}{9},\tfrac{2}{9},\tfrac{2}{9},\tfrac{1}{9},0\right)$ $\left(\tfrac{1}{6},\tfrac{1}{6},\tfrac{1}{6},\tfrac{1}{6},\tfrac{1}{6},\tfrac{1}{6}\right)$.

Theorem (for $\mathcal{E}_4$)

The set $\mathcal{E}_4$ consists of $(a_1, \ldots, a_8) \in \Omega_8$ satisfying

$$a_4 + a_5 + a_6 + a_7 \le \frac{1}{2} \le a_2 + a_3 + a_4 + a_5.$$

The extreme points of the sets are:

$$\left(\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, 0, 0, 0, 0\right), \left(\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4}, 0, 0, 0, 0, 0\right), \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, 0, 0, 0, 0, 0\right),$$

$$\left(\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, 0, 0, 0, 0\right), \left(\tfrac{1}{2}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, 0, 0, 0, 0\right), \left(\tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, 0, 0, 0\right),$$

$$\left(\tfrac{1}{2}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, 0, 0, 0\right), \left(\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, 0, 0\right), \left(\tfrac{3}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, 0, 0\right),$$

$$\left(\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{12}, \tfrac{1}{12}, 0\right), \left(\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{9}, \tfrac{1}{9}, \tfrac{1}{9}, 0\right), \left(\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, 0\right),$$

$$\left(\tfrac{3}{16}, \tfrac{3}{16}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, 0\right), \left(\tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, 0\right), \left(\tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}\right)$$

Some special extreme points

Note that vectors of the form $(\underbrace{1/k, \ldots, 1/k}_{k}, 0, \ldots, 0)$ may or may not be an extreme point of $\mathcal{E}_n$.

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Theorem

Let $(a_1, \ldots, a_{2n}) \in \Omega_{2n}$ satisfy $a_1 = \cdots = a_k = \frac{1}{k}$ for some positive integer k. Then $(a_1, \ldots, a_{2n}) \in \mathcal{E}_n$ if and only if $k \in \{n, 2n\}$, or there is $s \in \{1, \ldots, n-1\}$ such that

$$k = \frac{sn}{s+1} \qquad \text{or} \qquad k = 2n - \frac{sn}{s+1}.$$

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Question Is it possible to describe all the extreme points of $\mathcal{E}_n$?

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Question Is it possible to describe all the extreme points of $\mathcal{E}_n$?

If yes, we can describe all the inequalities (in theory).

Theorem (for $\mathcal{E}_5$)

The set $\mathcal{E}_5$ consists of $(a_1, \ldots, a_{10}) \in \Omega_{10}$ satisfying

$$a_7 + a_8 \leq \frac{1}{5} \leq a_3 + a_4$$

$$a_1 + a_8 + a_9 + a_{10} \leq \frac{2}{5} \leq a_1 + a_2 + a_3 + a_{10}$$

$$a_5 + a_6 + a_7 + a_{10} \leq \frac{2}{5} \leq a_1 + a_4 + a_5 + a_6$$

$$a_4 + a_7 + a_8 + a_9 \leq \frac{2}{5} \leq a_2 + a_3 + a_4 + a_7$$

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Theorem (for $\mathcal{E}_6$)

The set $\mathcal{E}_6$ consists of $(a_1, \ldots, a_{12}) \in \Omega_{12}$ satisfying

$$a_1 + a_{10} + a_{11} + a_{12} \le \frac{1}{3} \le a_1 + a_2 + a_3 + a_{12}$$

$$a_4 + a_9 + a_{10} + a_{11} \le \frac{1}{3} \le a_2 + a_3 + a_4 + a_9$$

$$a_7 + a_8 + a_9 + a_{10} \le \frac{1}{3} \le a_3 + a_4 + a_5 + a_6$$

$$a_1 + a_6 + a_8 + a_{10} + a_{11} + a_{12} \le \frac{1}{2} \le a_1 + a_2 + a_3 + a_5 + a_7 + a_{12}$$

Further research

The problem is open in general for $n \geq 7$. We only have some partial results.

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Theorem

Let $(a_1, \ldots, a_{2n}) \in \mathcal{E}_n$.

a. If $n = 2k + 1$, then $a_{3k+1} + a_{3k+2} \leq \frac{1}{n} \leq a_{k+1} + a_{k+2}$.
b. If $n = 2k$, then
   $a_{3k-2} + a_{3k-1} + a_{3k} + a_{3k+1} \leq \frac{1}{k} \leq a_k + a_{k+1} + a_{k+2} + a_{k+3}$

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Your comments and suggestions are most welcomed!

Thank you for your attention!